Published by Institute of Physics Publishing for SISSA/ISAS
Received: *August 22, 2002*
Revised: *January 20, 2003*
Accepted: *February 1, 2003*

Minkowski solution of Dyson-Schwinger equations in momentum subtraction scheme

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Abstract: Using the Green function integral representation the Dyson-Schwinger equations are solved directly in Minkowski space. Essential ideas of the spectral techniques are discussed and applied on two renormalizable models: the Yukawa theory with massive pseudoscalar meson and conventional spinor QED. Within the momentum subtraction procedure, the appropriate renormalization is performed analytically which leads to the usual dispersion formulation. The electron propagator obtained in this frame is compared with the solution of euclidean Dyson-Schwinger equation and with the perturbation theory results as well. The proposed method has the advantage of obtaining solutions in both the space- and *time*-like regimes of momenta. In addition,when the coupling constant increases we find some un-expected discrepancy between the euclidean and integral representation solutions. Especially, when the solutions of the original momentum DSE's exhibit the signal for confinement then the spectral approach gives no solution for Lehmann function of confined particle.

Keywords: Renormalization Regularization and Renormalons, Nonperturbative Effects.

Contents

1. Introduction

There are considerable interest in studies of strong coupling quantum field theories since many interesting phenomena, are believed have nonperturbative origins. In renormalizable field theory it is desirable to construct renormalization techniques which completely respect the symmetries of the underlying theory. In comparison with the conventional perturbation theory the nonperturbative treatment is more complicated and much effort is needed to perform renormalization procedure in a well-controlled manner. The fully regularization-independent method is proposed through the Dyson-Schwinger [1, 2] formalism, which is the main motivation of this study. The other motivation of the presented work is to avoid a constrains of euclidean metric. Therefore the main part of our calculation is performed directly in Minkowski space and we offer the obtained numerical solution of the Dyson-Schwinger equations (DSE's) at all regimes of momenta. If the euclidean analogue of solution is known and already present in the literature we perform the appropriate calculations of the euclidean DSE's and compare with Minkowski solution.

Contemporary application of DSE's in hadronic physics offers rigorous insight into the infrared domain of Quantum Chromodynamics [3, 4, 5] at zero temperature as well as they provide (DSE's) solid foundation for nonperturbative continuous approach at nonzero temperature and density [6]. Without the solution of DSE's it is difficult to reach any reasonable result on dynamical mass generation in the (Extended, Walking, ...) Technicolor models [7, 8, 9]. In these models (usually without higgses) the values of the various condensates and the masses of the Standard model particle content are obtained from the solution of the appropriate gap equations. In practice, the system of the DSE's is truncated by an approximation of the Green functions that were thrown away. In addition, the truncated set of DSE's must be renormalized and as it is usually required by the method, the loop integral must be regularized before proper renormalization step. When any S-matrix element is completed from the Green functions the definite results must be 1) renormgroup invariant, 2) gauge invariant 3) gauge fixing independent. All the three points are well understood in perturbation theory and it is desirable to build the similar nonperturbative method. However, the Pinch Technique [10, 11, 12] and the Background-Field Method [13] should offer the gauge fixing independent Green functions, but how to fulfill all three aforementioned points is not answered by fully satisfactory way in DSE's treatment. In this paper, we particularly concern on the points 1 and 2 mentioned above.

It is clear that the truncation of the DSE's system, if it is improperly performed, can violate gauge identity of the underlying gauge theory. In an abelian versions of gauge field theory (scalar and spinor quantum electrodynamic) this problem was already solved by at least two technically different way. Ball and Chiu [14] have derived the formula for the QED (truncated, proper) vertex that allows to close the DSE's system by a unique gauge covariant way (the lowest point Green functions satisfy Ward-Takahashi identities). The second successful approach is the longstanding method known under the name the 'Gauge Technique' [15, 16, 17, 18]. In this treatment the gauge covariant ansatze for the full (untruncated, i.e., not proper) vertex function is written in a terms of matter field spectral function. The technical advantage of Gauge Technique is that the resultant equation for fermion propagator is linearized in spectral function. Due to this simplification, the Gauge Technique's offer some solutions in compact analytical form. Of course, this linearization does not take place in the equation for the photon propagator and in fact this is one of the approach essential weakness: it cannot be true in general even for the case of electron propagator. Anticipate here, that no linearization does appear in this presented work and the appropriate integral equations are solved in their full form (they are third and second order integral equation). In the other side, the aforementioned linearization of Gauge Technique's obtained equation does not exclude their reliability in the soft coupling regime and we shall mention the Gauge Technique once again when we discussed infrared limit of electron propagator.

In the last decade relative large number of papers dealing with QED DSE's with the improved Ball-Chiu vertex has been written (mainly Ball-Chiu and/or Curtis-Pennington vertices were used). They studied the changes that follow from the improvement of the introduction of dominant transverse part vertex within the connection to dynamical mass generation [19, 20, 21], or they have been subjected to the various renormalization scheme used in its nonperturbative context [22, 23, 24, 25] (for the summary of the earlier works see [4]). The authors of the above mentioned works, have developed and successfully used some versions of numerical regularization. Particularly in the paper [25], the modified cut-off regularization method has been compared with the nonperturbative dimensional regularization scheme. This study exemplified the nonperturbative equivalence of different regularization-renormalization schemes. It also faces how much careful must be the appropriate numerical procedure in order to obtain the reliable physical results (without breaking gauge and Poincaré invariance of the underlying theory). In the proposed approach here, we extend this studies to the case of (direct) momentum subtraction (MOM) scheme where neither numerical regularization is used. Likewise in the perturbation treatment, the all appropriate loop integrals are subtracted at certain values of external momenta and being then finite they are integrated analytically. Such a procedure leads to the usual dispersion relations for renormalized proper functions. We use the Green function spectral representation which allows to convert momentum DSE's to the real equations for the spectral functions. At this point this approach is similar to the treatment used in the works [26, 27] and of course, to the one used in Gauge Technique studies. Note at this place, that similar method already has succeeded in the case of simple scalar models [28, 29, 30] where the on-shell renormalization scheme was pronounced.

It is worthwhile meaning that making some approximation the regularization independent and analytical answer can be obtained [25, 31, 32]. In very early stage of DSE's study there was made a sophisticated study [33] of spinor QED where the propagators entering calculation are taken as bare ones and the resulting vertex turns out to be a hypergeometric function in that case. Furthermore, in very special case (for massless fermion in rainbow and quenched Feynman-Fermi gauged QED) the authors of [34] found results analytically.

In our paper the method of solution will be illustrated at two model cases. The first of them is the Yukawa Theory (YT), i.e., the theory of fermions interacting with spinless boson. Furthermore, we choose this boson be a pseudoscalar. The Yukawa lagrangian reads explicitly:

$$\mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi - m_0\bar{\Psi}\Psi + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m_{0\phi}^2}{2}\Phi^2 - g_0 i\bar{\Psi}\gamma_5\Psi\Phi - h_0\Phi^4 \,. \tag{1.1}$$

Where m_0, $m_{0\phi}$ are bare masses of fermion and meson corresponding to the unrenormalized fields Ψ_f and Φ respectively and g_0, h_0 represent the unrenormalized values of coupling constants. For the sake of simplicity, the Yukawa vertex is modeled by its tree value thus making the coupling constant renormalization trivial in this case: $g = g_0$. In addition, it is assumed that $h \ll g$, therefore we also neglect the quartic mesons self-interaction. Remind here, that the quartic term in Φ is necessary due to the general requirement of renormalizability. The counter-term part has to cancel infinities appearing in the fermion loops contribution to scattering process $\Phi\Phi \to \Phi\Phi$. Here, the contribution from such process does not enter our DSE's due to their truncation mentioned above and an omission of $h_0\Phi^4$ represents self-consistent approximation.

The second model we employ is a conventional spinor QED. Although the approach is discussed for a general case of gauge covariant vertex the numerical results are presented

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for the quenched, rainbow approximation leaving the complete treatment for the later presentation. Approximation used in this article has the advantage of being simple enough, which makes it an excellent testing ground for the proposed minkowskian analysis. To specify remaining, the bare fermion mass m_o and gauge covariant linear fixing term are included explicitly in one flavor QED lagrangian. The so-called ladder approximation (the bare vertex is used, the name follows from the ladder approximation of the Bethe-Salpeter equation) is generally believed to be reliable in the Landau gauge only $\xi = 0$, and therefore we have used that gauge. Having all the numerical solutions stable and making comparison between euclidean and minkowskian results we found that they agree only when the coupling is small enough. Increasing the coupling constant the obvious discrepancy does appear. Approaching QED coupling $\alpha = e^2/(4\pi)$ to its critical value $\alpha = \pi/3$ we are leaving only with euclidean solution, whilst the spectral Minkowski equation tends to flaw.

The necessary analytical formulas are reviewed in the next section. It involves dispersion relation technique, its connection with MOM and derivation of unitary equations for DSE's. The section 4 is particularly devoted to the QED, where also the euclidean version of fermion DSE is reviewed. Numerical results are presented in the following section 5. The resulting non-nonperturbative solutions are endowed with attractive features like a correct analyticity and physical branching point. Some details of the calculation and of the numerical method are explained in the appendix A and the appendix B, respectively.

2. Spectral representation, analyticity and renormalization procedure

In the following sections we give overview of some basic facts about the Green function Spectral Representation (SR), Dispersion Relation (DR) technique and its relation to the renormalization.

The Lehmann representation [35, 36] for two body Green function can be derived from the Lorentz covariance and the quantum mechanical requirement on the positivity of spectral density. The relations (2.1) display the necessary SR's for appropriate propagators entering the calculation here:

$$\begin{aligned}
G(p^2) &= \int d\omega \frac{\bar{\sigma}(\omega)}{p^2-\omega+i\epsilon} = \frac{r_\phi}{p^2-m_\phi^2+i\epsilon} + \int d\omega \frac{\sigma(\omega)}{p^2-\omega+i\epsilon} \\
S_f(p) &= \int d\omega \frac{\not{p}\bar{\sigma}_v(\omega)+\bar{\sigma}_s(\omega)}{p^2-\omega+i\epsilon} = \frac{r_f}{\not{p}-m} + \int d\omega \frac{\not{p}\sigma_v(\omega)+\sigma_s(\omega)}{p^2-\omega+i\epsilon} \\
G^{\mu\nu}(p) &= \left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{p^2}\right) G_T(p^2) - \xi\frac{p^\mu p^\nu}{p^2} \\
G_T(p^2) &= \int d\omega \frac{\bar{\sigma}_\gamma(\omega)}{p^2-\omega+i\epsilon} = \frac{r_\gamma}{p^2+i\epsilon} + \int d\omega \frac{\sigma_\gamma(\omega)}{p^2-\omega+i\epsilon}\,.
\end{aligned} \tag{2.1}$$

Here G, S_f are the full propagators of particles with spin 0 and 1/2 respectively. $G^{\mu\nu}$ describes propagation of massless particle which corresponds to the gauge field. The longitudinal ξ-dependent part follows from the covariant linear gauge fixing $g_A = \delta(\xi - \partial_\mu A^\mu)$ of quantum action. The parameter r_i corresponds to residuum of i-particle propagators pole part. It should be shifted to the value $r_i = 1$ when the conventional on-shell renormalization scheme is pronounced. The full spectral functions $\bar{\sigma}_s$ are positive in non-gauge theory. The single particle contributions $r_i\delta(\omega - m_i^2)$ are integrated out from the full spectrum and it is assumed that remaining weight functions σ's in (2.1) are smoothed function and not complicated distributions at all.

It is notable in this place, that there is less formal proof of SR's for the simple Quantum field models. These models are represented by the scalar quantum field theories without derivative interaction. The existence of SR and the statement of its positivity is proved to the all orders of perturbation theory and even for an arbitrary $n-$points Green function. In this case, the so called Perturbation Theory Integral Representation (PTIR) was derived by Nakanishi [37]. Furthermore, it was shown that the PTIR is unique, which property appears to be very useful in some applications (see for instance [28, 30], and references herein).

Although the formal derivation of Lehmann representation is rather straightforward (see some standard textbooks [38, 39]), it necessarily breaks down when considering a theory without free particle asymptotic states, i.e. the theory with confinement. If the confinement takes place in a given theory then the particle can never be on mass shell and the appropriate propagator should not posses singularities at the real time-like axis of p^2. In the other words, the absence of Lehmann representation should be a good signal for confinement [40, 41]. Actually, as we will show in the section devoted to the strong coupling quenched QED, the disappearance of solution for Lehmann function rather sharply coincides with the transition to a confined phase.

As was mentioned, looking to the literature about DSE's the nonperturbative regularization-renormalization procedure is not so transparent as in the case of perturbative treatment. Restricting to the unconfined phase of given theory, the our task is whether the renormalization procedure should be performed analytically in an easy and transparent fashion as it is in the perturbative approach. First of all we must answer the question that naturally arises: what are the physical criteria, which will one to determined the solution of DSE's as a physically meaningful. In particular, the following is required for two point function:

1. The renormalized full Green function satisfies its own renormgroup equation $\gamma = \mu\frac{d}{d\mu}G(p^2, \mu)$, where gamma represents logarithmic differentiation of conventionally defined field strength renormalization constant Z (explicitly introduced later by eqs. (2.3)) $\gamma = -\mu\frac{dZ}{d\mu}$, i.e., the unrenormalized propagators $S_{f0}, G_0, G_0^{\mu\nu}$ have to be manifestly independent on the choice of renormalization scale.

2. The on-shell renormalization scheme (ORS), if it takes sense, is the only special choice of the renormalization scale $\mu = m$. Furthermore, the position of the pole $p^2 = m^2$ is renormgroup invariant quantity.

3. The proper Green functions (one particle irreducible diagram with truncated legs) are an analytical ones. Up to a subtracting polynom, the real and imaginary parts

of proper Green function is uniquely defined by a dispersion relation. In non-confining theory the propagators have their own Lehmann representation and they are Stjielts [50] transformable. The opposite should be true for confining phase of the theory: the absence of Lehmann representation is required.

4. The off-shell proper Green's functions calculated within DSE approach should admit renormalization. It implies that the possible maximum number of subtraction is two for self-energy and at most one for the possible triplet and quartic vertex renormalization. All the coefficients in the appropriate subtracting polynom should be absorb-able in the counter-term part of the original lagrangian.

5. As far as it is possible, the renormalization procedure should respect classical symmetry of the theory. Particularly, when dealing with gauge theory, the Green functions should satisfy Ward-Takahashi identities.

In the next subsections we review the general framework of analytical renormalization technique and explain the renormalization scheme that is actually used throughout this work. In the addition, we demonstrate that the Green function obtained by this technique satisfy all these requirement, including gauge covariance in QED case. For practical numerical evaluation in this article only, we do not play within full gauge covariant machinery that is necessary to satisfy the point (5). Instead of this we use the known fact that the bare vertex approximation is a satisfactory approximation when Landau gauge fixing $\xi = 0$ is imposed. Nevertheless, we introduce the most important relations already in this article, the numerical results and the detailed derivation of the dispersion relation beyond the rainbow and quenched approximation is relegated into the forthcoming paper.

2.1 Renormalization of DSE for Yukawa pseudoscalar

Here we start with the discussion of the spinless particle case. The unrenormalized version of DSE reads

$$\begin{aligned} G_0^{-1}(p^2) &= p^2 - m_0^2 - \widetilde{\Pi}_0(p^2) \\ \widetilde{\Pi}_0(p^2) &= i g_0^2 \int Tr \frac{d^4 l}{(2\pi)^4} \Gamma_{Y0}(p,l) S_0(l) \gamma^5 S_0(l-p) \, , \end{aligned} \tag{2.2}$$

where the index 0 represents bare quantities. To renormalize YT we conventionally introduce the renormalization functions with the appropriate counter-terms. Choosing some arbitrary renormalization scale μ^2 they are:

$$\begin{aligned} \Phi &= Z_\phi^{1/2} \phi_R \, , & \Psi &= Z_\Psi^{1/2} \Psi_R \, , \\ m_{\phi 0} &= Z_{m_\phi} m_\phi(\mu) \, ; & \delta_{m_\phi} &= m_0^2 - m(\mu)^2 \, , \qquad m_0 = Z_m m(\mu) \, ; \qquad \delta_m = m_0 - m(\mu) \, , \\ g_0 &= \frac{Z_g^{1/2}}{Z_\Psi Z_\phi^{1/2}} g \, , & h_0 &= \frac{Z_h^{1/2}}{Z_\phi^2} h \, , \end{aligned} \tag{2.3}$$

where Z_Φ $(Z_\Psi)$is a renormalization boson (fermion) field-strength constant, Z_m is the mass renormalization constant, m_0 is a bare mass while $m(\mu)$ represents renormalized mass.

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Furthermore, Z_h and Z_g represent the renormalization constant of quartic Φ^4 and triplet $\bar{\Psi}\Psi\Phi$ vertex respectively, i.e. for instance we can write for the renormalized proper triplet vertex:$\Gamma_Y(p,p-k) = Z_g\Gamma_{Y0}(p,p-k)$. First of all we renormalize the propagators with respect to the field renormalization only i.e. $G_0 = Z_\phi G$, $S_0 = Z_\psi S$. For this purpose we multiply the unrenormalized DSE (2.2) by the constant Z_ϕ. A simple algebra gives

$$
\begin{aligned}
G^{-1}(p^2) &= Z_\phi(p^2 - m_0^2) - \Pi_0(p^2) \\
\Pi_0(p^2) &= ig^2 \int Tr\frac{d^4l}{(2\pi)^4}\Gamma_Y(p,l)S_f(l)\gamma^5 S_f(l-p)\,,
\end{aligned}
\tag{2.4}
$$

where the proper vertex Γ_Y satisfies its own DSE. To truncate the system of DSE we make most simple approximation $\Gamma_Y(p,l) = \gamma^5$.

Since the pseudoscalar particle Φ requires quadratically divergent mass renormalization and logarithmically divergent field strength renormalization the relation between renormalized and unrenormalized self-energy function must be of the form

$$
\Pi_0(p^2) = \Pi_0(\mu^2) + \Pi_0'(\mu^2)(p^2-\mu^2) + \Pi(\mu;p^2)
\tag{2.5}
$$

where the renormalized self-energy satisfies double subtracted DR

$$
\Pi(\mu;p^2) = \int d\omega \frac{\rho_\phi(\omega)}{(p^2-\omega+i\epsilon)}\left[\frac{p^2-\mu^2}{\omega-\mu^2}\right]^2 .
\tag{2.6}
$$

Of course, from the rel. (2.5) we obtain the standard receipt for calculation of the propagator in MOM scheme

$$
\begin{aligned}
G^{MOM}(\mu,p^2) &= \left(p^2 - m^2(\mu) - \Pi(\mu;p^2)\right)^{-1} \\
\Pi(\mu;p^2) &= \Pi_0(p^2) - \Pi_0(\mu^2) - \frac{d}{dp^2}\Pi_0(p^2)\Big|_{p^2=\mu^2}(p^2-\mu^2)
\end{aligned}
\tag{2.7}
$$

From eq. (2.5), one can readily see that the unrenormalized value $\Pi_0(\mu^2)$ corresponds to the dominant part of the mass renormalization constant and the derivative $\Pi_0'(\mu^2)$ corresponds to the field-strength renormalization, explicitly we have

$$
\begin{aligned}
Z_\phi &= 1 + \Pi_0'(\mu) \\
Z_{m_\phi}^2 &= Z_\phi^{-1}\left(1 - \frac{\Pi_0(\mu^2) - \Pi_0'(\mu^2)\mu^2}{m^2(\mu)}\right).
\end{aligned}
\tag{2.8}
$$

Note that in perturbative context this scheme is sometimes referred as BPHZ renormalization scheme and the appropriate subtractions of the first terms in Taylor expansion of Π_0 is called (Bogoliubov) R-operation (without explicitly performed regularization). In the next text we do not write explicit dependence on μ if it is not necessary, also we drop out the labeling MOM since no other renormalization scheme is used throughout this article.

Employing the well known functional identity for distributions

$$
\frac{1}{x'-x+i\varepsilon} = P.\frac{1}{x'-x} - i\pi\delta(x'-x)
\tag{2.9}
$$

we see that the function $\pi\rho(p^2)$ represents the absorptive (imaginary) part of renormalized self-energy $\Pi(p^2)$ as well as of the unrenormalized one. The last statement is unrelated with the question of (in)finiteness of the counter-terms, since such renormalization procedure has nothing directly with the presence of infinities and can be consistently applied to a theories which are UV convergent. The detailed derivation of (2.10) is relegated to the appendix A. Here we simply review the result:

$$\begin{aligned}
\rho_\phi(\omega) = & \left(\frac{g}{2\pi}\right)^2 \Bigg[r_f^2 \frac{\omega}{2}\sqrt{1-\frac{4m^2}{\omega}}\Theta(\omega-4m^2) + \\
& + m r_f \int_{(m+m_\phi)^2}^{\infty} d\beta \left(\frac{1}{2}(\omega - m^2 - \beta)\sigma_v(\beta) + m\sigma_s(\beta)\right) X(\omega; m^2, \beta) + \\
& + \int_{(m+m_\phi)^2}^{\infty} d\alpha \int_{(m+m_\phi)^2}^{\infty} d\beta \Bigg(\frac{1}{2}(\omega - \alpha - \beta)\sigma_v(\alpha)\sigma_v(\beta) + \\
& + \sigma_s(\alpha)\sigma_s(\beta)\Bigg) X(\omega; \alpha, \beta)\Bigg]
\end{aligned} \tag{2.10}$$

where the function X can be expressed through the triangle function λ by the following way

$$X(a; b, c) = \frac{\lambda^{1/2}(a; b, c)}{a}\Theta(a - (\sqrt(b) + \sqrt(c))^2) \tag{2.11}$$

and m is pole mass of fermion.

For the practical purpose it is useful to know the relation between the renormalized mass $m_\phi(\mu)$ and physical (pole or resonant) mass $m_\phi = m_\phi(m\phi)$ that is conventionally defined as $G^{-1}(m_\phi^2) = 0$. From its definition it reads:

$$m_\phi^2(\mu) = m_\phi^2 + \int d\omega \frac{\rho_\phi(\omega)}{\omega - m_\phi^2}\left[\frac{m_\phi^2 - \mu^2}{\omega - \mu^2}\right]^2 . \tag{2.12}$$

From the above discussion it follows that the requirements written in the beginning of this section are clearly satisfied. We end this section by making an interesting comment. It is worthwhile meaning that there is no need to solve explicitly renormgroup equation to see μ dependence explicitly. Due to the algebraic simplicity of the DR in the MOM scheme we can immediately recognize that the inverse of the full propagators G renormalized at two different scales μ, μ' just differ by finite polynom $a_{\text{fin.}} + b_{\text{fin.}}p^2$. Written explicitly, the relation reads

$$\begin{aligned}
G^{-1}(\mu, p^2) &= a_{\text{fin.}} + b_{\text{fin.}}p^2 + G^{-1}(\mu', p^2) \\
a_{\text{fin.}} &= \frac{\Pi(\mu;\mu'^2) + \Pi(\mu';\mu^2)}{\mu^2 - \mu'^2} = \int d\omega \frac{\rho_\phi(\omega)(\mu'^2 - \mu^2)^2}{(\omega - \mu'^2)(\omega - \mu^2)^2} \\
b_{\text{fin.}} &= m^2(\mu') - m^2(\mu) + \frac{\mu'^2\Pi(\mu';\mu^2) + \mu^2\Pi(\mu;\mu'^2)}{\mu'^2 - \mu^2} \\
&= m^2(\mu') - m^2(\mu) + \int d\omega \frac{\rho_\phi(\omega)\omega(\mu'^2 - \mu^2)}{(\omega - \mu'^2)(\omega - \mu^2)^2} .
\end{aligned} \tag{2.13}$$

2.2 Renormalization of DSE for Yukawa fermion

The extension to the fermion case can be proceed by very similar way. The DSE for fermion propagator is conventionally parameterized like:

$$S_f^{-1}(\not p) = Z_\Psi[\not p - m_0] - \Sigma_0(p) \,, \tag{2.14}$$

where m_0 represents bare mass of fermion and the renormalization constant Z_ψ was already introduced in (2.3). It is convenient to split unrenormalized self-energy to its Dirac vector and scalar part

$$\Sigma_0(p) = \not p a_0(p^2) + b_0(p^2) \,. \tag{2.15}$$

The scalar functions a_0, b_0 can be easily identified from the explicit expression for fermion self-energy

$$\Sigma_0(\not p) = -ig^2 \int \frac{d^4l}{(2\pi)^4} \Gamma_Y(p,l) S_f(l) \gamma^5 G((p-l)^2) \,. \tag{2.16}$$

In YT (and of course also in QED) the self-energy is only logarithmically divergent and one subtraction is sufficient to make the scalar functions a_0, b_0 finite. The renormalized self-energy then reads

$$\Sigma(\mu, p) = \not p a(\mu, p^2) + b(\mu, p^2) \,, \tag{2.17}$$

where the subtraction leads to the following DR's for renormalized a, b

$$\begin{aligned} a(\mu,p^2) &= a_0(p^2) - a_0(\mu^2) = \int ds \frac{\rho_v(s)(p^2-\mu^2)}{(p^2-s+i\varepsilon)(s-\mu^2)} \\ b(\mu,p^2) &= b_0(p^2) - b_0(\mu^2) = \int ds \frac{\rho_s(s)(p^2-\mu^2)}{(p^2-s+i\varepsilon)(s-\mu^2)} \,. \end{aligned} \tag{2.18}$$

Then the renormalized version of (2.14) can be written like

$$S_f^{-1}(p) = A(\mu)[\not p - m(\mu)] - \Sigma(\mu, p) \,. \tag{2.19}$$

where $m(\mu)$, $(A(\mu))$ represents renormalized fermion mass (coefficient of $\not p$) fixed at the scale μ. As it is usually in DSE treatment the relation (2.19) can be equivalently rewritten into the form:

$$\begin{aligned} S^{-1}(p) &= A(p^2)\not p - B(p^2) \\ A(p^2) &\equiv A(\mu) - a(\mu, p^2) \\ B(p^2) &\equiv A(\mu)m(\mu) + b(\mu, p^2) \,, \end{aligned} \tag{2.20}$$

where again we do not write explicitly dependence on μ in the renormalized functions S, A, B. For the absorptive parts $\mathrm{Im}\, A = \mathrm{Im}\, a = \pi\rho_v; \mathrm{Im}\, B = \mathrm{Im}\, b = \pi\rho_s$ we can find the following results:

$$\rho_v(\omega) = \frac{-g^2}{(4\pi)^2}\left[r_f r_\phi X_1(\omega; {m_\phi}^2, m^2) + m r_\phi \int_{(m+m_\phi)^2}^{\infty} d\alpha \sigma_v(\alpha) X_1(\omega; m_\phi^2, \alpha) + \right. \tag{2.21}$$

$$\left. + r_f \int_{4m^2}^{\infty} d\beta \rho_\phi(\beta) X_1(\omega; \beta, m^2) + \int_{4m^2}^{\infty} d\beta \int_{(m+m_\phi)^2}^{\infty} d\alpha \rho_\phi(\beta)\sigma_v(\alpha) X_1(\omega; \beta, \alpha)\right].$$

$$\rho_s(\omega) = \frac{g^2}{(4\pi)^2}\left[r_\phi r_f m X(\omega; m_\phi^2, m^2) + r_\phi \int_{(m+m_\phi)^2}^{\infty} d\alpha \sigma_s(\alpha) X(\omega; m_\phi^2, \alpha) + \right. \tag{2.22}$$

$$\left. + \int_{4m^2}^{\infty} d\beta \rho_\phi(\beta) X(\omega; \beta, m^2) + \int_{4m^2}^{\infty} d\beta \int_{(m+m_\phi)^2}^{\infty} d\alpha \rho_\phi(\beta)\sigma_s(\alpha) X(\omega; \alpha, \beta)\right].$$

where the function X has been introduced earlier and $X_1(x; y, z) = \lambda^{1/2}(x; y, z)(x - y + z)/(2x^2)\Theta(x - (\sqrt(y) + \sqrt(z))^2)$. The detailed derivation of eqs. (2.21), (2.22)is presented in the appendix A.

In the DSE's studies it is a good habit to evaluate the renormgroup invariant dynamical mass which is conventionally defined like

$$M(p^2) = \frac{B(p^2)}{A(p^2)}\,. \tag{2.23}$$

Since the on shell mass $m \equiv m(m)$ is usually the one which is the best known experimentally, it is useful to introduce the relation between m and the one renormalized at μ. From its definition $S(m)^{-1} = 0$ we have

$$m = A(\mu)^{-1}[m(\mu) + a(\mu; m^2)m + b(\mu; m^2)] \tag{2.24}$$

where in this paper we explicitly choose

$$A(\mu) = 1 \tag{2.25}$$

and for practical purpose $m(\mu)$ is fixed through (2.24) such that $m = 1$.

Imposing the renormalization condition on S at two different scales μ, and μ' we can again recognize that the inverse of the fermion propagator differs by certain finite piece $c_{\text{fin.}} + \not{p} d_{\text{fin.}}$, here we only note that the coefficients $c_{\text{fin.}}, d_{\text{fin.}}$ can be expressed through the functions $a, b(\mu; \mu'^2)$ and $a, b(\mu'; \mu^2)$.

2.3 DSE for $\sigma' s$ — the unitary equations

The derivation of the DSE for Lehmann weights is presented in this section. Evaluating the imaginary part of the trivial identity $1 = G(p^2)G(p^2)^{-1}$ with SR (2.1) used for G and using DR form of DSE $G^{-1} = p^2 - m_\phi(\mu) - \Pi(\mu; p^2)$ to express the inverse of G gives

$$\sigma(\omega)(\omega - m_\phi^2(\mu)) = r_\phi \frac{\rho_\phi(\omega)}{\omega - m_\phi^2} + \{\rho_\phi * \sigma\}\,, \tag{2.26}$$

where the symbol $\{\rho_\phi * \sigma\}$ represents following principal value integral:

$$P.\int dx \frac{\rho(s)\sigma(x)+\sigma(s)\rho(x)\frac{(s-\mu^2)^2}{(x-\mu^2)^2}}{s-x} \, . \tag{2.27}$$

The pseudoscalar propagator residuum

$$r_p = \lim_{p^2 \to m_\phi^2} \frac{p^2-m_\phi^2}{G^{-1}(p^2)} \tag{2.28}$$

is most easily evaluated through the appropriate DR for self-energy

$$r_p = \frac{1}{1-\Pi'(m_\phi^2)} = \left[1 - \int d\omega \frac{\rho_\phi(\omega)(m_\phi^2-\mu^2)(m_\phi^2+\mu^2-2\omega)}{(m_\phi^2-\omega)^2(\omega-\mu^2)^2}\right]^{-1} \, . \tag{2.29}$$

It is not surprising that the equation (2.26) looks particularly simply in on mass-shell renormalization scheme:

$$\sigma(\omega) = \frac{\rho_\phi(\omega)}{(\omega-m_\phi^2)^2} - \frac{\{\rho_\phi * \sigma\}}{\omega-m_\phi^2} \, . \tag{2.30}$$

(i.e. Roughly pronounced: the perturbative Lehmann function of spinless particle corresponds to the absorptive part of self-energy divided by the factor $(\omega - m_\phi^2)^2$ and the principal value integral is the high order effect.)

The fermion case can be treat by a very similar way. From the fermion DSE we can obtain for the residuum

$$r_f = \lim_{\not p \to m} \frac{\not p - m}{S^{-1}(p)} = [1 - a'(\mu; m^2)m - b'(\mu; m^2) - a(\mu; m^2)]^{-1} \, . \tag{2.31}$$

where we explicitly used the renormalization condition (2.25). Furthermore, writing the trivial identity $S^{-1}S - 1 = 0$ in a suitable form:

$$\begin{aligned} &\left[\frac{r_f}{\not p - m} + \int d\omega \frac{\not p \sigma_v(\omega) + m\sigma_s(\omega)}{p^2-\omega+i\varepsilon}\right] \times \\ \times &\left[\not p - m(\mu) - \int ds \frac{(\not p \rho_v(s) + \rho_s(s))(p^2-\mu^2)}{(p^2-s+i\varepsilon)(s-\mu^2)}\right] = 1 \end{aligned} \tag{2.32}$$

then we can easily arrived to the relations between Lehmann weights σ's and the absorptive parts of self-energy, i.e., ρ's. Projecting the obtained result by $\frac{Tr}{4\pi p^2}$ and $\frac{Tr}{4\pi m p^2}\not p$ leads to the coupled set of integral equations

$$\begin{aligned} \sigma_v(\omega) &= \frac{f_1 + m(\mu) f_2}{\omega - m^2(\mu)} \, ; \qquad \sigma_s(\omega) = \frac{m(\mu) f_1 + \omega f_2}{\omega - m^2(\mu)} \\ f_1 &\equiv r \frac{\omega \rho_v(\omega) + m\rho_s(\omega)}{\omega - m^2} + \omega[\sigma_v * \rho_v] + [\sigma_s * \rho_s] \\ f_2 &\equiv r \frac{m\rho_v(\omega) + \rho_s(\omega)}{\omega - m^2} + \omega[\sigma_s * \rho_v] + [\sigma_v * \rho_s] \, . \end{aligned} \tag{2.33}$$

where ω is (positive in our metric) time-like momentum $\omega \equiv p^2$ and where we have used the abbreviation for the real functional:

$$[\sigma * \rho] = P. \int_{m^2}^{\infty} dx \frac{\rho(s)\sigma(x) + \sigma(s)\rho(x)\frac{s-\mu^2}{x-\mu^2}}{s-x} . \qquad (2.34)$$

where $P.$ stands for principal value integration.

Note also, that the fixed renormalization implies the unique determination of the propagator residuum. Henceforth, its value must be also obtainable without on-shell differentiation of the self-energy function. Indeed, the alternative formula but fully equivalent to (2.31) can be simply derived from (2.32). It reads

$$r = \left(1 + m\frac{m - m(\mu)}{m(\mu)m - \mu^2}\right)\left(1 + \int d\omega \frac{\mu^2\sigma_v(\omega) - m(\mu)\sigma_s(\omega)}{\omega - \mu^2}\right) . \qquad (2.35)$$

3. DSE's in quantum electrodynamic

After the more general introduction we describe minkowskian formalism which is necessary in QED DSE's treatment. For this purpose we used the conventions already established in the last two previous sections, the differences that appear are emphasized. In the end of this section we also review the euclidean analog of the fermion DSE, its solution serves us for comparison of the methods.

In general, there is no isolated pole but propagator singularity coincides with the branching point. The analytical structure of QED fermion propagator was the subject of the initial study [42]. The authors of the work [42] converted the integral gap equation (for its explicit form see rel. (3.15) in the text bellow) to the non-linear differential equation which has been then solved by graphical method. In addition, they proceed the backward Wick-rotation of their equation to the time-like regime of momenta. Consequently, they did not find zero in the inverse electron propagator (for a real p^2). This unexpected disappearance of physical branch point leads the author to the conclusion that confinement should exists even in the abelian gauge theory. Having the ultraviolet cut-off Λ fixed to the some finite but large value ($\Lambda \gg M(0)$), the authors of [42] identify the coupling of the phase transition to be exactly the one related to the dynamical mass generation, i.e. $\alpha = \pi/3$ in Landau gauge.

In fact, the claim that the abelian gauge theory has a confinement phase sounds strange from the conventional wisdom based on the continuum abelian gauge theory. It is clear enough, that the electrons must be a free particles when the coupling is small enough. The most recent work can divert this unreliability. It was already shown in the paper [43] that the abelian confinement should exist only at the above certain value of strong coupling constant, probably where the photon reveals the dynamical mass. Making the quenched approximation, the photon mass generation can not be answered, but the corresponding disappearance of the physical branch point is observed and confirmed in our study. In agreement with our expectation, this transition is observed for the value $\alpha \simeq 1$ of QED coupling constant which coincides (up to the numerical accuracy) with the failure of our spectral approach.

The later analyze [44] founded that the electron propagator (in the ladder approximation) develops singularity for the all couplings. The results of the paper [45] support partially the conclusion made in [44] (but, not for all the couplings): The electron propagator has not only one real branch point, as it is physically expected, but it also embodies two other complex conjugated singularities (the position of them entails troubles with the analytical continuation, see the discussion in [4]). To conclude, note that it is generally believed, that this unexpected pole's complexity should vanishes for the exact solution (particularly when the coupling is small enough). In fact, the existence of the complex branch points is questionable and the appropriate answer depends on the approximation employed. Note that, it is not necessary in the contradiction with the result of us and with the earlier study of Fukuda-Kugo [42]. It would be only in the case if the pure real pole solution does not vanish for the coupling constants large enough.

In the spectral approach used throughout this work, the dominant part of the propagator is driven by the real pole part: $r_f(p^2-m^2)^{-1}$ and the changes coming from the interaction are involved in continuous part of Lehmann spectrum

$$\frac{\not p\sigma_v(\omega)+\sigma_s(\omega)}{p^2-\omega+i\epsilon} \tag{3.1}$$

which may but need not to be analytical at the branch point $p^2=m_+^2$. Recall at this place the Gauge Technique (zeroth order iteration) result [16] where in class of covariant gauges the infrared behavior reads

$$S(\not p)\simeq\frac{1}{\not p-m}\left(\frac{m^2}{p^2-m^2}\right)^{\frac{\alpha}{2\pi}(\xi-3)}\,. \tag{3.2}$$

We actually see that the interaction partially suppressed pole singularity for those gauge fixing parameters that are less then Yennie gauge ($\xi=3$ in Yennie gauge).

Furthermore, in contrast to the YT, we are now dealing with the gauge theory and the appropriate renormalization scheme should respect the gauge identity. Mainly the vacuum polarization tensor should be transverse

$$q^\mu\Pi_{\mu\nu}(q)=0 \tag{3.3}$$

and the proper photon-electron-electron vertex should satisfy Ward-Takahashi identity

$$S^{-1}(p)-S^{-1}(l)=(p-l)_\mu\Gamma^\mu(p,l) \tag{3.4}$$

which uniquely determined the longitudinal part of the vertex [14]

$$\begin{aligned}\Gamma^\mu_L(p,l)\,=\,&\frac{\gamma^\mu}{2}\left(A(p^2)+A(l^2)\right)+\\&+\frac{1}{2}\frac{(\not p+\not l)(p^\mu+l^\mu)}{p^2-l^2}\left(A(p^2)-A(l^2)\right)-\frac{p^\mu+l^\mu}{p^2-l^2}\left(B(p^2)-B(l^2)\right).\end{aligned} \tag{3.5}$$

Using the dispersion relation for the function A,B

$$\begin{aligned}A(p^2)\,&=1-\int d\alpha\frac{\rho_v(\alpha)(p^2-\mu^2)}{(p^2-\alpha+i\varepsilon)(\alpha-\mu^2)}\\B(p^2)\,&=1+\int d\alpha\frac{\rho_s(\alpha)(p^2-\mu^2)}{(p^2-\alpha+i\varepsilon)(\alpha-\mu^2)}\end{aligned} \tag{3.6}$$

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we obtain the integral representation for Γ^μ_L

$$\begin{aligned}\Gamma^\mu_L(p,l) &= \gamma^\mu\left(1-\int d\omega\frac{\rho_v(\omega)}{\omega-\mu^2}-\int d\omega\frac{\rho_v(\omega)}{p^2-\omega}-\int d\omega\frac{\rho_v(\omega)}{l^2-\omega}\right)-\\ &-\frac{(p^\mu+l^\mu)(\not p+\not l)}{2}\int d\omega\frac{\rho_v(\omega)}{(p^2-\omega)(l^2-\omega)}-\\ &-(p^\mu+l^\mu)\int d\omega\frac{\rho_s(\omega)}{(p^2-\omega)(l^2-\omega)}\,. \end{aligned} \tag{3.7}$$

From this expression one can read immediately see that the longitudinal part of QED vertex is free of any kinematical singularities and we also see that the only coefficient of γ^μ has explicit renormalization point dependence. Furthermore, we can note here that this should be the whole μ explicit dependence of the full vertex $\Gamma^\mu=\Gamma^\mu_L+\Gamma^\mu_T$, since its transverse part (satisfying $(p-l).\Gamma_T=0$) must be finite (for more details of (perturbative) MOM renormalization scheme used in QED see [46]).

Substituting the gauge covariant vertex (3.5) into the DSE's for fermion and photon propagator we obtain two closed equations that can be solved numerically after the renormalization (see most recent paper [31] and references therein) Adopting rather standard notation: let the constants Z_1,Z_2 and Z_3 represent the vertex, fermion wave function and photon wave function respectively then the renormalization proceeds by the standard way: the Dirac and scalar function $a_0(\mu)$ and $b_0(\mu)$ in unrenormalized Σ_0 must be absorbed in Z_2 and Z_m renormalization constant by the same way as it is happen in the case of Yukawa fermion propagator. Furthermore the unrenormalized vacuum polarization $\Pi(\mu)$ should be absorbed in the renormalization constant Z_3 and the infinity of Γ^μ is canceled against the constant $1-Z_1$. The renormalization scale used to renormalize vertex and the electron propagator should be the same due to the WTI (3.4) that requires $Z_1=Z_2$. Also note here, that we do not explicitly indicate the renormalization constants dependence on their renormalization scales (i.e., $Z_i=Z_i(\mu)$).

In our MOM 'imaginary part analysis' we must self-consistently obtain DR (3.6) for fermion function, further we should obtain DR for renormalized photon vacuum polarization

$$\begin{aligned}\pi^{\mu\nu}_R(\mu,q) &= q^2\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\pi^{MOM}_R(\mu^2;q^2)\\ \pi^{MOM}_R(\mu^2;q^2) &= \int_0^\infty d\omega\frac{q^2-\mu^2}{(q^2-\omega+i\epsilon)(\omega-\mu^2)}\rho(\omega)\,, \end{aligned} \tag{3.8}$$

where its absorptive part reads

$$\pi\rho(\omega)=\frac{\alpha_{QED}}{3}\left(1+\frac{2m^2}{\omega}\right)\sqrt{1-\frac{4m^2}{\omega}}\,\Theta(\omega-4m^2)+O(e^4) \tag{3.9}$$

with $\alpha_{QED}=e^2/(4\pi)$, e is a charge of electron, m is on shell electron mass. Without making some other approximation the full treatment with the DSE's leads to the evaluation of large number loop integrals and it also requires a careful numerical treatment due to the presence of Landau ghost. The complete solution of this problem is relegated to the forthcoming paper [47]. Here we exploit the aforementioned property of Landau gauge where the rainbow truncation is satisfactory approximation. Although it is not so difficult, we do not go beyond quenched approximation, since rainbow approximation would violate not only WTI (3.4) but also the condition of transversality (3.3).

Using the bare vertex and quenched approximation the fermion DSE reads

$$Z_2[\not p - m_0] - ie^2 \int \frac{d^4k}{(2\pi)^4} G_0^{\mu\nu}(k)\gamma^\mu S(p-k)\gamma^\nu = 0 \,. \tag{3.10}$$

and straightforward calculation gives $a(\mu;p^2) = 0$. This entails equality $A(\mu,p^2) = 1$ for all square of momenta [48] if the condition $A(\mu) = 1$ is imposed and also the unitary equations are notable simplified

$$\begin{aligned} \sigma_v(\omega) &= \frac{f_1 + m(\mu) f_2}{\omega - m^2(\mu)} ; & \sigma_s(\omega) &= \frac{m(\mu) f_1 + \omega f_2}{\omega - m^2(\mu)} \\ f_1 &\equiv r\frac{m\rho_s(\omega)}{\omega - m^2} + [\sigma_s * \rho_s] ; & f_2 &\equiv r\frac{\rho_s(\omega)}{\omega - m^2} + [\sigma_v * \rho_s] \end{aligned} \tag{3.11}$$

where the absorptive part of the renormalized self-energy $\mathrm{Im}\,\Sigma(p) = b(\mu;p^2) = \pi\rho_s(p^2)$ is given by

$$\rho_s(\omega) = -3\left(\frac{e}{4\pi}\right)^2 \left[m\left(1 - \frac{m^2}{\omega}\right) + \int_{m^2}^{\omega} d\alpha \sigma_s(\alpha)\left(1 - \frac{\alpha}{\omega}\right)\right] . \tag{3.12}$$

The notation and abbreviations used here is the same as the ones in section devoted to the Yukawa model. Some details of the derivation of rel. (3.12) is given in the part b) of the appendix A.

In order to make a careful and constructive comparison between Minkowski approach presented in this work and the standard euclidean formulation we should compare with some known euclidean results presented in the literature [42, 22]. In space-like region the eq. (3.10) is transfered to

$$\begin{aligned} B(x) &= m_0 + \frac{3\alpha}{4\pi}\int_0^\infty dy K(x,y)\frac{B(y)}{y + B^2(y)} \\ K(x,y) &= \frac{2y}{x + y + \sqrt{(x-y)^2}} \end{aligned} \tag{3.13}$$

where Wick rotation (i.e. we assume its validity here) and angle integration have been done and where we have used $Z_2 = 1$, $x \equiv p_E^2 = -p^2$, $y \equiv k_E^2 = -k^2$. The renormalized equation then reads

$$\begin{aligned} B(\zeta,x) &= m(\zeta) + \frac{3\alpha}{4\pi}\int_0^\infty dy V(\zeta,x,y)\frac{B(\zeta,y)}{y + B^2(\zeta,y)} \\ V(\zeta,x,y) &= K(x,y) - K(\zeta,y) \end{aligned} \tag{3.14}$$

where ζ is square of renormalization scale $-\mu^2$. Choosing the scale ζ to be zero and further scaling the renormalized $m(\zeta)$ mass as $m(0) = 1$ leads to the particularly simple expression for (3.14)

$$B(0,x) = 1 + \frac{3\alpha}{4\pi}\int_0^x dy\left(\frac{y}{x} - 1\right)\frac{B(0,y)}{y + B^2(0,y)} \tag{3.15}$$

(for 'massive photon' case see eq. (3.14) in the work [42]). Stressed here, that due to the

Landau gauge the solution of eq. (3.14) is only very slightly deviating from the solution with Curtis-Pennington vertex implemented [22]. From the paper [25] the asymptotic behavior of dynamical mass is known analytically. Henceforth, any trustable solution of electron DSE must behaves at ultraviolet like

$$M(p_E^2) = M(\zeta)\left(\frac{p_E^2}{\zeta}\right)^s . \tag{3.16}$$

Note also, that $A(\mu^2;p^2) = 1$ at ultraviolet for solution with Curtis-Pennington vertex (again due to the Landau gauge, remind that it is unit for all p^2 in our approximation).

The results of the last equation and the eq. (3.14) for different choice of ζ are presented in the next section. After the continuation of eq. (3.15) to the time-like regime and adopting a principal value integration we also estimate the behavior of a real part of the function $B(p^2 > 0)$.

4. Numerical solution and results

4.1 Yukawa theory

The resulting coupled nonlinear integral equations (2.33), (2.26) for the functions $\sigma_{v,s}$ and σ_ϕ require the knowledge about the value of physical masses (2.24), (2.12), the propagators residua (2.29), (2.35) and the complete knowledge of the absorptive parts of self-energies (2.10), (2.21), (2.22). All the functions in the previous statement depend on the $\sigma's$, and after the making some choice of the renormalized masses the system of seven integral equations to be solved is fully closed. They have been solved by the method of numerical iteration which seems to be particularly useful for this purpose. The one loop perturbation theory result is used as the zeroth order of this iteration. Then, several hundred of iteration steps have been proceed to achieve a stable and accurate solution. For purpose of numerical integration we choose the gaussian quadrature method. Taking a reasonable number of the integration mash points and adopting the principal value integration described in the appendix B, then the whole numerical procedure is rather stable against the changing of mesh points density and their number as well as.

The all presented results for YT are evaluated for the zero value of renormalization scale $\mu^2 = 0$ which choice is common for propagators corresponding with both the particle of YT. The physical mass is usually the best known experimentally and we face that this is our case. Let us assume that experimentalists (living in our toy model world) found their values: $m_\phi = 0.15m$, $m = 1$. In this case, we are enforced to calculate $m(\mu)$ and $m_\phi(\mu)$ from the rels. (2.24), (2.12) which does not cause any troubles in our iteration method. The above described procedure has clear numerical advantage: the branching points lie at the values of momenta which are fixed at each iteration step, but it has also disadvantage: the comparison with some results obtained in the euclidean formalism is not so straightforward (note, $m(\mu)$ is fixed for some space-like μ^2). To make the most accurate comparison with euclidean result, we should fix the renormalization scale and the renormalized masses at the same values that are used in the euclidean DSE's. We prefer the first scheme in YT since the appropriate euclidean solution is not published elsewhere (but we use the second approach in QED case).

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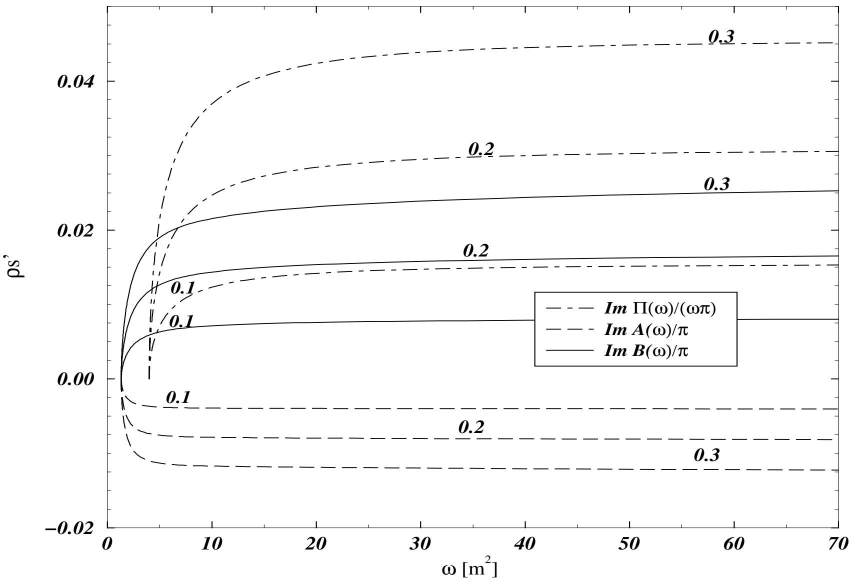

Figure 1: The spectral functions for Yukawa model calculated for the values $0.1; 0.2; 0.3$ of coupling strength $\lambda = g^2/4\pi$. The lines with the threshold $(m + m_\phi)^2 = 1.15^2 m^2$ correspond with the Dirac (negative lines) and scalar (positive lines). The Lehmann weights for pseudoscalar propagator have the appropriate thresholds at $4m^2$ and they are always positive.

Figure 1 shows the absorptive part of self-energies, i.e., the functions ρ_v, ρ_s for fermion and ρ_ϕ for pseudoscalar. They are plotted for the following values: $\lambda = 0.1, 0.2, 0.3$ of the coupling strength defined as $\lambda = g^2/4\pi$. For a large momenta the pseudoscalar spectral function grows linearly with the square of momenta that is an unpleasant consequence of the quadratic divergence in Π_0. Due to this, what we actually plot is the rescaled function $\rho_\phi(x)/x$. All the functions start to be non zero from their perturbative thresholds- $\mathrm{Im}\, A(s), \mathrm{Im}\, B(s)$ from $s = (m_\phi + m)^2$ and $\mathrm{Im}\,\Pi(s)$ from $s = 4m^2$. Because the negative parity of the field Φ the functions ρ_s, ρ_ϕ (and σ_s, σ_ϕ) are positive while ρ_v (σ_v) is negative. The quadratic divergence of the unrenormalized self-energy Π leads to the quadratic dependence (+corrections) of the renormalized Π. The quantity $Q^{1/2}$ of dimension $[\text{mass}]^1$ defined with the help of the inverse of propagator $G^{-1}(p^2) = p^2 - Q(p^2) = p^2 - m_\phi^2(\mu) - \Pi(\mu; p^2)$ can not be called the dynamical boson mass, since Q becomes negative above the certain value of euclidean momenta. The same happens at the time-like axis for $\mathrm{Re}\, Q(p^2)$ (while $\mathrm{Im}\, Q(p^2) = \pi\rho_\phi$ is positive as it is clear from figure 1). The appropriate behavior of the function Q is displayed in figure 2. The momentum dependence of the fermion functions A, B is dominated by the (perturbative) logarithm of p^2. Indeed, the Yukawa fermion propagator is almost given by its free form corrected by small perturbation. Figure 3 and figure 4 displays the momentum dependence of the function A, B for space-like momenta and time-like momenta respectively. Although, the smallness of the ratio m_{phi}/m is motivated by the desired enhancement of fermion self-energy and subsequent suppression of pseudoscalar one, nevertheless, we still see that the fermion-meson loop becomes irrelevant perturbative contribution for all studied couplings. In agreement with the previous statement the dynamical mass of the fermion very sightly deviate from its pole value at

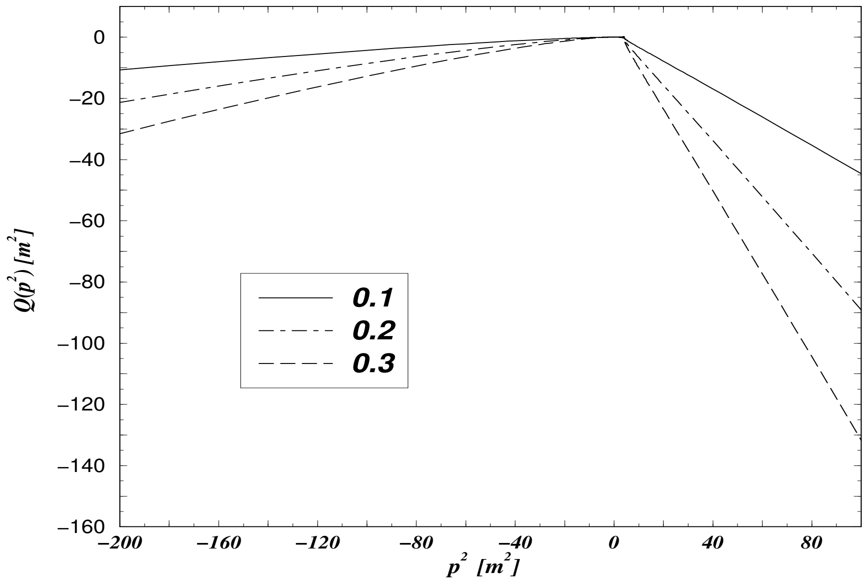

Figure 2: The self-energy of pseudoscalar meson $Q(p^2) = m_\phi^2(\mu) + \Pi_R(\mu^2, p^2)$ for various coupling strengths λ of Yukawa interaction. The only real part is displayed above the threshold $4m^2$.

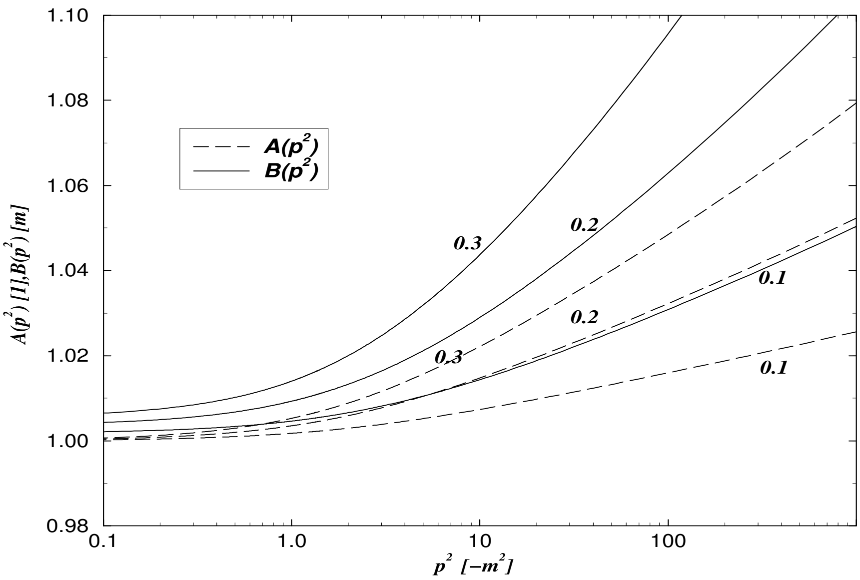

Figure 3: Yukawa fermion functions $A(p^2), B(p^2)$ at space-like regime of momentum.

all p^2. Therefore, we display the dynamical mass $M = B/A$ in the infrared domain in the figure 5, because this is the only interesting regime of momenta. The appearance of the critical coupling λ_c is the consequence of the quadratic momentum dependence of pseudoscalar self-energy. Its value slightly depends on the numerical cut-off Λ used in our unitary equation, noting that within our numeric ($\Lambda^2 = 10^7 m^2$) the solution of unitary equations sharply starts to fail at $\lambda_c \simeq 0.3$.

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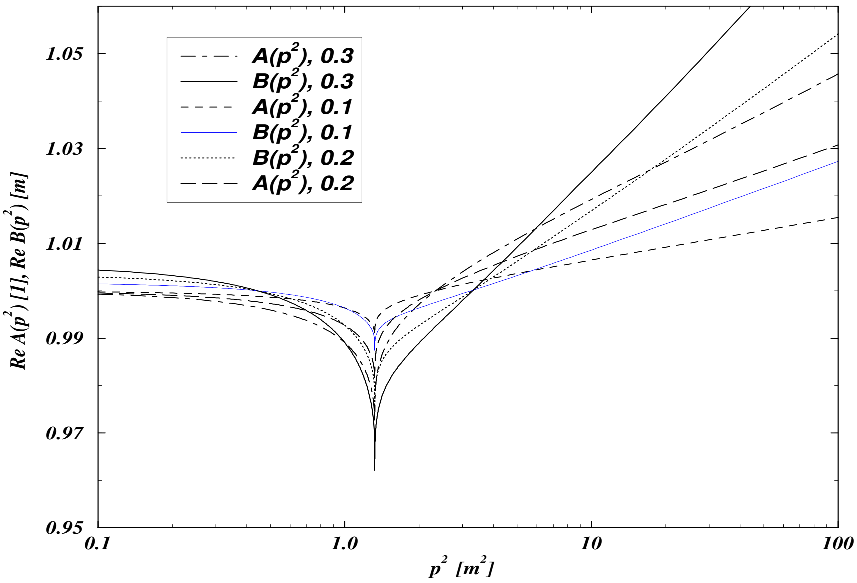

Figure 4: Yukawa fermion functions $A(p^2), B(p^2)$ at time-like regime of momentum.

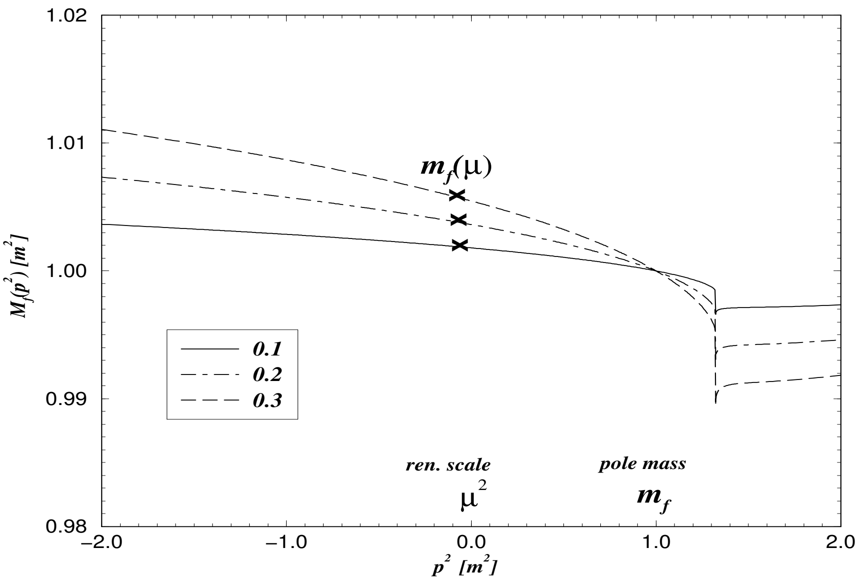

Figure 5: The renormalization scale invariant dynamical mass of Yukawa fermion. The renormalized mass $m(0)$ is fixed at $\mu = 0$ such that the pole mass is $m_f = 1$ for all the couplings $\lambda = 0.1; 0.2; 0.3$.

4.2 QED fermion propagator

In contrast to the model discussed previously, the strong coupling QED is less driven by the perturbation theory. Recall here the famous paper [49] dealing with scalar electrodynamics. It is the simplest model where the phenomena of radiative corrections induced spontaneous symmetry breaking occurs. Although there is no clear evidence for such a phenomena in spinor QED, it remains an instructive challenge. Before presentation of our numerical solution, let us recollect the main results:

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- Comparing the electron propagator obtained by solving the Unitary Equations (UE's) (3.11) with the propagator calculated in the euclidean formalism we find that they never exactly agree. The exception is the case of very small coupling $\alpha \simeq 10^{-2}$, where both approaches seem to be equivalent (up to the numerical errors). Stressed here, that in this case, they are almost indistinguishable form the perturbation theory result.

- Previous statement is valid also in the case when the small photon mass λ is introduced. The obtained results are then slightly changed quantitatively and the disagreement discussed in 1. is somehow small (the euclidean$\times$Minkowski results numerically agree, only when $\lambda \simeq m$, which is the case, we are not interested in). It leads us to the conclusion, that this discrepancy do not fully follows from the masselness of the photon and from the appropriate subsequent coincidence of the electron propagator singularity with the branch point. Note, that the last property is not simply fulfilled for nonzero λ.

- The Minkowski (spectral) solution exists only for subcritical regime of α_{QED}, whilst, as it is well known the euclidean solution can be easily find for both the sub- and super-critical couplings. To be more precise, we have not found any solution of UE's when the coupling was larger then $\alpha_{\text{c.s.}} = 0.95$ (c.s.=critical spectral). Further, making an estimate of the momentum DSE solution (not assuming spectral decomposition) we have found strong evidence for confinement for the coupling larger then $\alpha = 0.91 \pm 0.1$. (Remind,that the critical coupling in QED is defined such that $B(p^2) = 0$ for $m_0 = 0$, (chiral symmetric phase) for $\alpha < \alpha_c$ and the solution for $B(p^2)$ is non-zero and finite for $\alpha > \alpha_c$; $m_0 = 0$ when momentum cut-off Λ is implemented (without finite Λ the function B tends to diverge). Its value is known from the quenched, rainbow study, where $\alpha_c = \pi/3$, more sophisticated studies do not deviate significantly from this value).

The UE's (3.11) complemented by the equations for the absorptive part ρ_s and for the residuum have been solved by the method of iterations. In order to have an infrared behavior under the control and in order to see the effect of the vector boson mass we also implement the small mass parameter λ into the photon propagator:

$$G_o^{\mu\nu}(k) = \frac{-g^{\mu\nu} + k^\mu k^\nu/k^2}{k^2 - \lambda^2 + i\varepsilon} \, . \tag{4.1}$$

Restricting to the small photon mass case, the Dirac part coefficient function A is approximated as $A(p^2) = 1$, i.e. as it would be in the massless case. The expression for function ρ_s is then slightly change (see appendix A), noting here that the limit $\lambda \to 0$ is smoothly achieved as it follows from the appropriate relations (A.25) presented in the appendix A. The solutions with several different photon masses have been obtained. Then, the dynamical mass $M = B$ at space-like momenta is calculated from the absorptive functions $\rho_s, \sigma_s, \sigma_v$, which are primary solutions here. We can see from the figure 6 that the effect of

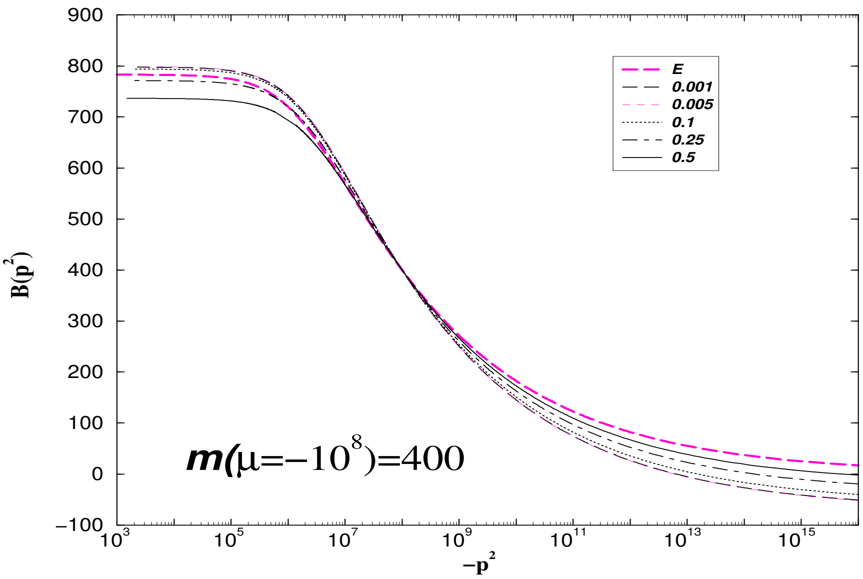

Figure 6: The dynamical mass of the electron in massive photon QED as they have been obtained by solving modified unitary equtaions. The various lines are labeled by the mass of the photon λ which is written in the units of an on-shell electron mass. All solutions have the coupling value $\alpha = 0.6$, renormalization point $\mu^2 = -10^8$, and renormalized mass is $m(\mu) = 400$. The results for two smallest λ are not distinguishable. Thin long dashed line labeled by the letter E represents the solution obtained in euclidean formalism. In this case, photon was exactly massless.

the small photon mass is almost irrelevant even if the coupling is relatively large ($\alpha = 0.6$). The space-like renormalization scale is chosen to be large $\mu^2 = -10^8$ when compared with renormalized mass $m^2(\mu) = 400^2$, which choice corresponds with the euclidean solution already presented in the work [24]. The appropriate euclidean solution of eq. (3.14) is added for the comparison. The same (in)dependence on the parameter λ is observed for any choice of $\mu, m(\mu)$. In order to achieve good numerical stability of UE's some very small photon mass is always used in. The presented results in this work are calculated with $\lambda = 10^{-3}m$.

Solutions were obtained for the Fukuda-Kugo equation (FKE) (3.15) for the couplings from 0.01 to 2.0. The expected [42, 22] damping of dynamical mass to its negative values was observed for supercritical couplings $\alpha > \pi/3$. Within the using of the same renormalization point $M(0) = 1$ the UE's were solved. The resulting solution for M is plotted for the following couplings: $0.2, 0.4, 0.6$ and compared with the FKE solution. All these solutions are plotted against the space-like momentum in figure 7. From this we can see the relatively large discrepancy between the results of FKE and UE's. Clearly, there are some hidden flaw at least in the one of two approaches employed. We have no apparent explanation for this observed discrepancy and we speculate in the Conclusion. Making great numerical effort we have found the main cause numerically. Let us assume that the residuum and the principal value integral integrations that appear in UE are somehow over-estimated due to the infrared enhancement of the Lehmann weights. Let us introduce small coupling constant dependent infrared cutoff $c{*}m^2$ to the UE's in a way that modified

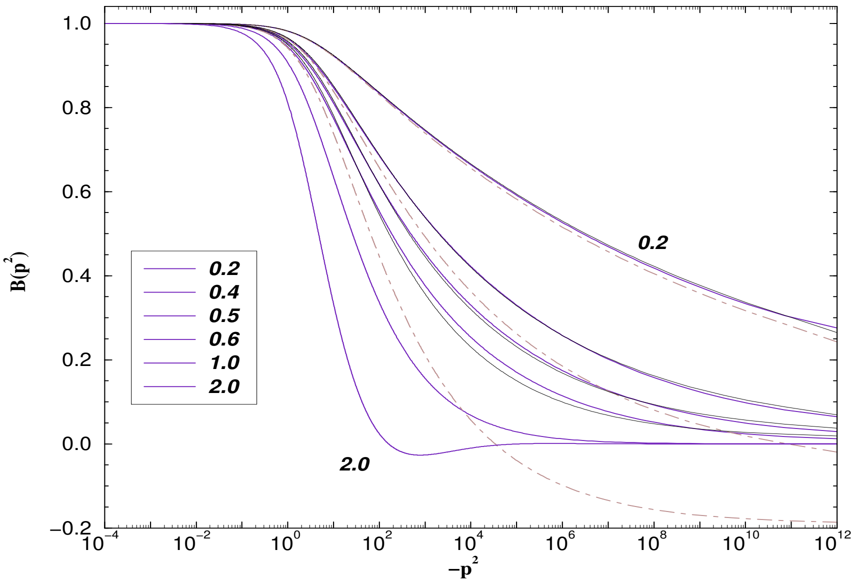

Figure 7: The electron dynamical mass $M(p^2)$ for space-like regime of momentum. Six dotted lines represent the euclidean solutions of momentum DSE. They are labeled by the value of coupling constant at given order (from up to down). The solutions of unitary equations (dot dashed lines) and the ones of modified unitary equations (solid thin lines) are added for the coupling $\alpha = 0.2; 0.4; 0.6$.

UE's now sounds:

$$\begin{aligned}
\sigma_v(\omega) &= \frac{f_1 + m(\mu) f_2}{\omega - m^2(\mu)}; \qquad \sigma_s(\omega) = \frac{m(\mu) f_1 + \omega f_2}{\omega - m^2(\mu)} \\
f_1 &\equiv r \frac{m \rho_s(\omega)}{\omega - m^2} + [\sigma_s * \rho_s]\Theta(\omega - m^2(1+c)) \\
f_2 &\equiv r \frac{\rho_s(\omega)}{\omega - m^2} + [\sigma_v * \rho_s]\Theta(\omega - m^2(1+c)) \\
r &= \left(1 + m\frac{m - m(\mu)}{m(\mu) m - \mu^2}\right)\left(1 + \int d\omega \frac{\mu^2 \sigma_v(\omega) - m(\mu)\sigma_s(\omega)}{\omega - \mu^2}\right)\Theta(\omega - m^2(1+c)) \\
c &= \frac{\alpha}{2\pi},
\end{aligned} \tag{4.2}$$

noting that taking $c = 0$ then the modified unitary equation's (MUE's) (4.2) clearly reproduce the originally UE's. Looking at the solution of MUE's we see that their solution is in the reasonable agreement with the solutions of FKE. In the next we limit ourselves to the subcritical values of coupling constant where the infrared cut-off $c * m^2$ is small (compared with the pole mass). The remainder discrepancy shows us nothing else but how our guess (4.2) was (or was not) wrong. Figure 8 displays the dynamical mass behavior at the time-like regime obtained for the same choice of renormalization mass as previous. The maximum corresponds correspond with the on mass-shell mass point. Note that their existence is in contradiction with the assumption of confinement [42]. Although, we are pretty sure that the solution must look like as in this figure, in order to assure, we solve the time-like continued equation of Fukuda-Kugo. However, we are not able to find the exact and fully complex solution numerically, nevertheless, we made a sophisticated estimate of

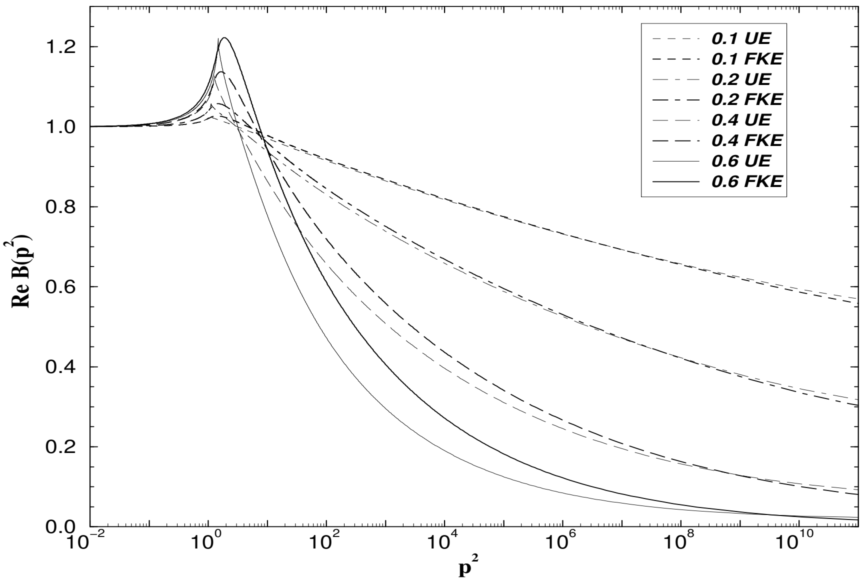

Figure 8: The electron dynamical mass $M(p^2) = B(p^2)$ for time-like regime of momentum and renormalization choice $m(0) = 1$. The thick lines represent euclidean solutions obtained by principal value integration of the Fukuda-Kugo equation. The thin lines correspond with the solutions of modified unitary equations (labeled by UE).

a real part of $B(p^2)$ by the principal value integration. It is achieved by the standard numerical introduction of a finite epsilon, here it is chosen to be a fraction of pole mass and presented solutions are obtained with the choice $\epsilon = 0.03m^4$, i.e., the singular integrand is approximated like

$$P.\frac{1}{z - B^2(z)} \to \frac{z - B^2(z)}{(z - B^2(z))^2 + \epsilon} \tag{4.3}$$

where z is a time-like momentum $z = p^2$.

The soft coupling results are added to the figure 8 and fully agree with our expectation: the branch point always exists because the mass function $B^2(x)$ always cuts the graph of the function $y(x) = x$. This statement is valid for QED with subcritical couplings $\alpha < 0.915$. At most, we expect about ten percentage deviation from the exact value of α_{crit} which should be a consequence of the numerical method weakness. Furthermore, we can see from the figure 8 that FKE solution for $\mathrm{Re}\, B$ reasonably agree with the solutions of MUE's, especially when the coupling is small enough. But we have rather different situation in strong coupling QED. The maximum of the mass function disappear and the mass function never cuts the graph $\sqrt{x}$. *The supercritical coupling quenched QED is a confining theory, the electron propagator has not a branch point and is free of any singularities.* In this case, the ϵ introduced above can be safely limited to the zero value and we actually omited it when FKE was solved for $\alpha > \alpha_{\mathrm{crit}}$. Henceforth, we can conclude that the supercritical couplings ladder approximation of electron DSE has purely real solutions for all momenta $p^2 \in (-\infty, +\infty)$. The results are plotted in figure 9 where we also add some subcritical solutions for better comparison. The subcritical solutions have been obtained with the help

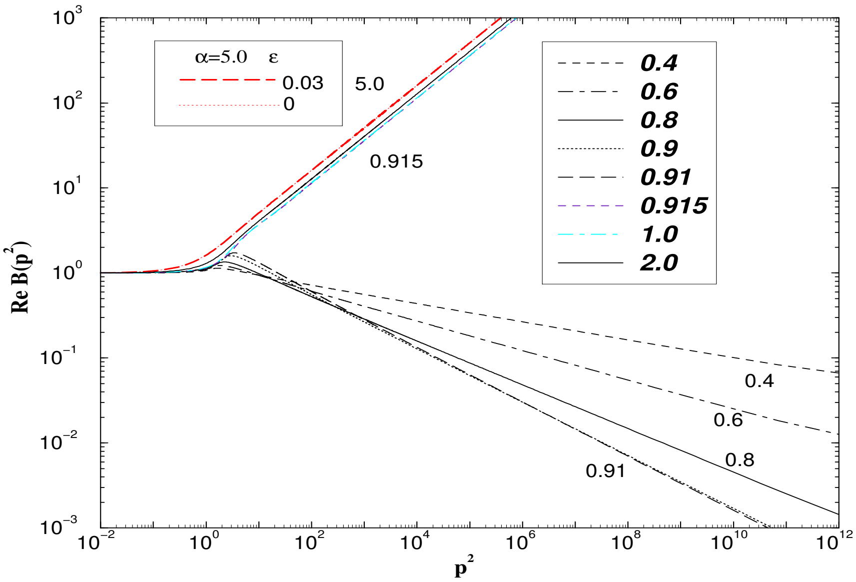

Figure 9: The same as in the previous figure, but for stronger couplings. The results are obtained from solution of Fukuda-Kugo equation only. Signal for confinement is apparent for the coupling $\alpha > 0.915$. The lines represent a good estimate of true FKE solution for subcritical couplings but they must be rather exact for the supercritical couplings solutions (for the meaning of ϵ see (4.3)).

of time-like continued FKE and there is no spectral solutions presented in this figure (the value of infrared cut-off c makes the MUE's untrusted).

The program using the euclidean equation (3.14) was run with several coupling α from 0.2 to 8.0 (!), and with renormalization choice $\mu^2 = -100$, and $m(\mu) = 10$. Whenever α coincides with the one published in [22], then the results here and the ones obtained in [22] are numerically identical. Again, we see for supercritical couplings, there is a small region where the dynamical mass is negative. When the coupling constant is extremely large (say $\alpha = 8$) the negative damping becomes a relatively fast oscillation around the zero mass axis. This feature should be in agreement with [22] but in a subtle disagreement with the paper [42], where the oscillation appears to be purely positive. The solution of B in space-like regime is presented in figure 10. The solutions of MUE's and of the euclidean momentum DSE are compared only in the space-like regime. The MUE's solutions are presented in in figure 11 for time-like momenta. We do not solve the appropriate time-like continued momentum DSE, Because the scale invariance of the theory, we do not solve the momentum DSE at time-like regime, up to the scaling the behavior of the solution must be the same as in the case discussed above.

An unexpected feature of the spectral equations is that for any set $\{\alpha > 0.9, \mu^2, m(\mu)\}$ the absorptive self-energy becomes oscillating around the zero axis. This behavior was verified to be insensitive to number of grid points, but was not stable against the infrared cut-off introduced above. In fact, the positions of minima's and maxima's are slowly walking when the cut-off c is varied by hand. The significance of this for spectral representation and its connection to QED is not completely understood. One possibility is that it may be signal the failure of Lehmann representation. The examples of this feature is shown in

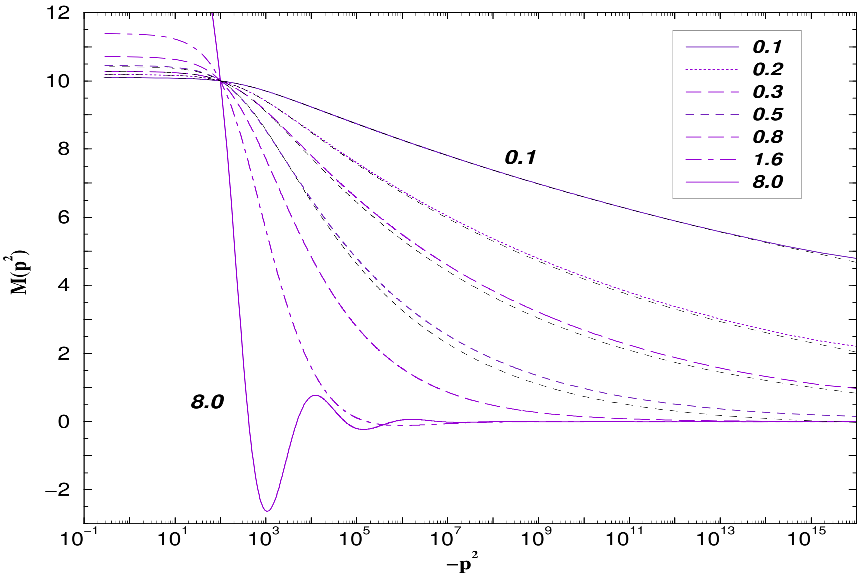

Figure 10: Space-like solutions for electron propagator. All solutions have the renormalization point $\mu^2 = -100$, and renormalized mass $m(\mu) = 10$ and couplings is varied from 0.1 to 8.0. Note that there are several zero-crossing when the coupling is large enough. The solutions of modified unitary equations (dashed thin lines which are closed to their euclidean counter-partners) are added for the lowest couplings $\alpha = 0.1, 0.2, 0.3, 0.5$.

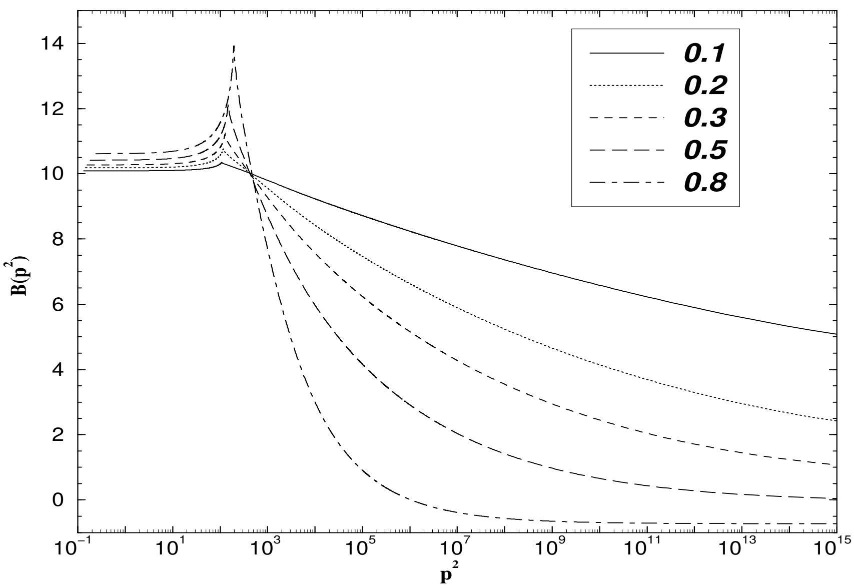

Figure 11: Time-like solution for electron propagator as they have been obtained from modified unitary equations. The renormalization choice is the same as in the previous figure, i.e. $m(-100) = 10$.

figure 12. It should be emphasized here, that all the results for $M(p^2)$ obtained by solving of MUE's and presented in this work have been calculated from non-oscillating, smoothed (soft coupling constant) ρ_s's.

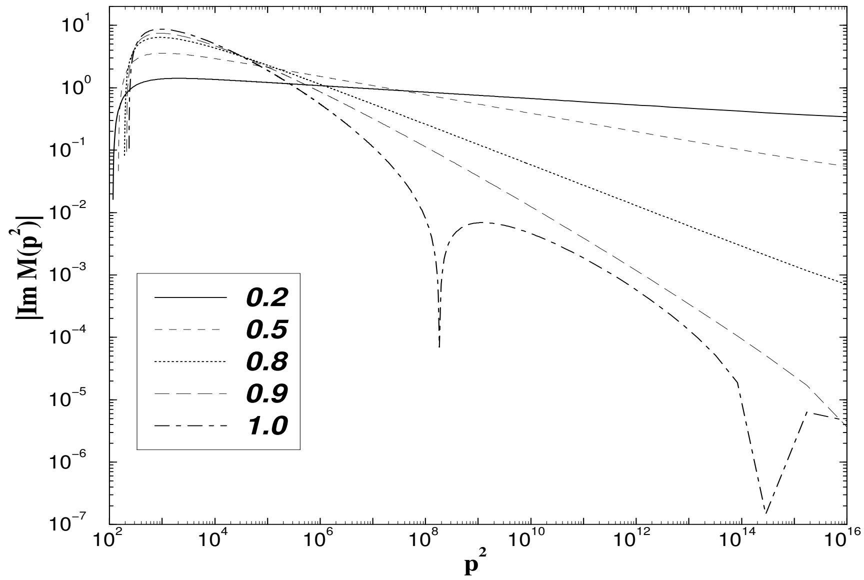

Figure 12: The absolute values of absorptive part of electron self-energy. In order to obtain the result for $\alpha = 1$, we used the infrared cut-off value $0.2m$ in the modified unitary equations treatment (m is a pole mass, here always $m \simeq m(-100) = 10$, for instance $m = 14$ for $\alpha = 0.8$.

5. Conclusions and outlook

In this article we rewrite the Dyson-Schwinger equations for propagators to the equations for their Lehmann weights. The possibility to do this, is based on the assumption of the existence of Green function integral representation which should be valid in nonperturbative regime. The resulting set of equations is manifestly regularization independent and the full Green functions fulfill the statement of renormalization scale invariance. They have been solved without some unwanted linearization or angle approximation. This is the novelty and significant advantage in comparison with the usual manner commonly used in the literature. The developed approach has been tested on two models. The QED electron propagator has been compared with the results obtained in conventional euclidean formalism. We found certain discrepancy between these approaches which appears to be rather large when the coupling approaches its critical value. Furthermore, the solution obtained within the help of Lehmann representation is fully absent for supercritical couplings regime of QED. In this case the branch point vanishes and the electrons (positrons) must be confined.

Those interested in applications of this technique may request a copy of the fortran computer codes from the author at the given e-mail address

Acknowledgments

Author is very grateful to R. Dermíšek and A. Cieplý for the careful reading of the manuscript. I also thank to Prof. R. Delbourgo who point my attention on some existing literature. This research was supported by GA ČR under Contract n. 202/00/1669.

A. Evaluation of the off-shell dispersion relations

In this appendix we derive the appropriate DR's for the proper Green functions of YT. As a first step we analyze the fermion self-energy.

A.1 Yukawa fermion self-energy

As follows from (2.16) the general expression which has to be considered has the following structure:

$$\begin{aligned}\Sigma_0(\not p) &= -ig^2\int\frac{d^4l}{(2\pi)^4}\frac{\gamma^5(C_a\not l+C_b)\gamma^5}{(l^2-\alpha+i\epsilon)((l-p)^2-\beta+i\epsilon)}\\ &= -ig^2\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\frac{-C_a\not p(1-x)+C_b}{[l^2+p^2x(1-x)-\alpha x-\beta(1-x)+i\epsilon]^2}\,,\end{aligned}\tag{A.1}$$

where the whole prefactors are absorbed into the formal symbols $C_{a,b}$. Written in the terms of continuous functions $\sigma_{v,s}$ and Lehmann weight σ for pseudoscalar, the symbol C_a and C_b are identified like

$$\begin{aligned}C_a &= \int d\alpha\left\{\delta(\alpha-m^2)+\sigma_v(\alpha)\right\}\int d\beta\left\{\delta(\beta-m_\phi^2)+\sigma(\beta)\right\}\\ C_b &= \int d\alpha\left\{\alpha^{1/2}\delta(\alpha-m^2)+\sigma_s(\alpha)\right\}\int d\beta\left\{\delta(\beta-m_\phi^2)+\sigma(\beta)\right\}.\end{aligned}\tag{A.2}$$

Noting that they should be collocated in front of the fractions in (A.1).

Observing the Dirac structure, the unrenormalized functions a_0, b_0 can be simply identified. After making subtractions (2.18) we can arrive to their renormalized forms

$$\begin{aligned}a(\mu;p^2) &= ig^2C_a\int_0^1 dxdy\int\frac{d^4l}{(2\pi)^4}\frac{(1-x)^2x2(p^2-\mu^2)}{[l^2+(p^2-\mu^2)x(1-x)y+\mu^2x(1-x)-O+i\epsilon]^3}\\ b(\mu;p^2) &= -ig^2C_b\int_0^1 dxdy\int\frac{d^4l}{(2\pi)^4}\frac{2(1-x)x(p^2-\mu^2)}{[l^2+(p^2-\mu^2)x(1-x)y+\mu^2x(1-x)-O+i\epsilon]^3}.\end{aligned}\tag{A.3}$$

where $O=\alpha x-\beta(1-x)$. Integrating over the loop momentum and making substitution $y\to\omega$

$$\omega=\mu^2-\frac{\mu^2}{y}+\frac{O}{x(1-x)y}\,,\tag{A.4}$$

we obtain the DR's

$$\begin{aligned}a(\mu;p^2) &= -C_a\left(\frac{g}{4\pi}\right)^2\int_0^1 dx\int_{\frac{O}{x(1-x)}}^{\infty}d\omega\frac{1-x}{\omega-\mu^2}\frac{p^2-\mu^2}{(p^2-\omega+i\epsilon)}\\ b(\mu;p^2) &= C_b\left(\frac{g}{4\pi}\right)^2\int_0^1 dx\int_{\frac{O}{x(1-x)}}^{\infty}d\omega\frac{p^2-\mu^2}{\omega-\mu^2}\frac{1}{(p^2-\omega+i\epsilon)}\,.\end{aligned}\tag{A.5}$$

Using the following definition of B functions

$$\begin{aligned}X(\omega;\alpha,\beta) &= \int_0^1 dx\Theta\left(\omega-\frac{O}{x(1-x)}\right)\\ X_1(\omega;\alpha,\beta) &= \int_0^1 dxx\Theta\left(\omega-\frac{O}{x(1-x)}\right)\end{aligned}\tag{A.6}$$

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we can rewrite DR(A.5) into the more familiar form:

$$
\begin{aligned}
a(\mu;p^2) &= -C_a\left(\frac{g}{4\pi}\right)^2\int d\omega\frac{X_1(\omega;\beta,\alpha)(p^2-\mu^2)}{(\omega-\mu^2)(p^2-\omega+i\epsilon)}\\
b(\mu;p^2) &= C_b\left(\frac{g}{4\pi}\right)^2\int d\omega\frac{X(\omega;\alpha,\beta)(p^2-\mu^2)}{(\omega-\mu^2)(p^2-\omega+i\epsilon)}\,,
\end{aligned}
\tag{A.7}
$$

where we made use of the relation

$$
X(\omega;\alpha,\beta)-X_1(\omega;\alpha,\beta)=X_1(\omega;\beta,\alpha)\,. \tag{A.8}
$$

Explicit integration in (A.6) gives the following formulas for B functions

$$
\begin{aligned}
X(\omega;\alpha,\beta) &= \frac{\lambda^{1/2}(\alpha,\omega,\beta)}{\omega}\Theta\left(\omega-(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})^2\right)\\
X_1(\omega;\alpha,\beta) &= \frac{\lambda^{1/2}(\alpha,\omega,\beta)}{2\omega}\left[1+\frac{\beta-\alpha}{\omega}\right]\Theta\left(\omega-(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}})^2\right)\,.
\end{aligned}
\tag{A.9}
$$

where λ is the triangle Khallén function $\lambda=(\omega-\alpha+\beta)^2-4\omega\beta$.

After the explicit introduction of the prefactors $C_{a,b}$ we can write down the absorptive part of self-energy functions a,b:

$$
\begin{aligned}
\rho_a(\omega) = \frac{-g^2}{(4\pi)^2}\Bigg[&R_f r_\phi X_1(\omega;m_\phi{}^2,m^2)+m r_\phi\int_{(m+m_\phi)^2}^{\infty}d\alpha\sigma_v(\alpha)X_1(\omega;m_\phi^2,\alpha)+\\
&+r_f\int_{4m^2}^{\infty}d\beta\rho_\phi(\beta)X_1(\omega;\beta,m^2)+\\
&+\int_{4m^2}^{\infty}d\beta\int_{(m+m_\phi)^2}^{\infty}d\alpha\rho_\phi(\beta)\sigma_v(\alpha)X_1(\omega;\beta,\alpha)\Bigg]\,,
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
\rho_b(\omega) = \frac{g^2}{(4\pi)^2}\Bigg[&r_\phi r_f m X(\omega;m_\phi^2,m^2)+r_\phi\int_{(m+m_\phi)^2}^{\infty}d\alpha\sigma_s(\alpha)X(\omega;m_\phi^2,\alpha)+\\
&+r_f\int_{4m^2}^{\infty}d\beta\rho_\phi(\beta)X(\omega;\beta,m^2)+\\
&+\int_{4m^2}^{\infty}d\beta\int_{(m+m_\phi)^2}^{\infty}d\alpha\rho_\phi(\beta)\sigma_s(\alpha)X(\omega;\alpha,\beta)\Bigg]\,.
\end{aligned}
\tag{A.11}
$$

As follows from the properties of function B the various terms in (A.10), (A.11) start to be nonzero from different values of ω. This knowledge is particularly useful when calculated numerically. Indeed after the inspection of $X(\omega;\alpha,\beta)$ we can find the threshold at $\omega=(\sqrt{\alpha}+\sqrt{\beta})$. For instance the subtresholds values $(m_\phi+m)^2;(2m_\phi+m)^2;9m^2;(m_\phi+3m)^2$ correspond to the terms in eq. (A.11) at given order.

Yukawa pseudoscalar self-energy Let us find the DR for pseudoscalar self-energy Π. The loop integral relevant to our discussion has the form:

$$\begin{aligned}\widetilde{\Pi}(p^2) &= i\int\frac{d^4l}{(2\pi)^4}Tr\frac{\gamma_5[\not{l}\bar{\sigma}_v(\alpha)+\bar{\sigma}_s](\alpha)\gamma_5[(\not{l}-\not{p})\bar{\sigma}_v(\beta)+\bar{\sigma}_s(\beta)]}{(l^2-\alpha+i\epsilon)((l-p)^2-\beta+i\epsilon)}\\ &= 4i\int\frac{d^4l}{(2\pi)^4}\left[\frac{-\bar{\sigma}_v(\alpha)\bar{\sigma}_v(\beta)}{((l-p)^2-\beta+i\epsilon)}+\right.\\ &\qquad\left.+\int_0^1 dx\frac{(-\alpha+p^2(1-x))\bar{\sigma}_v(\alpha)\bar{\sigma}_v(\beta)+\bar{\sigma}_s(\alpha)\bar{\sigma}_s(\beta)}{l^2+p^2x(1-x)-O}\right], \qquad (A.12)\end{aligned}$$

where the shift $l\to l+p(1-x)$ is made at the second integral and $O=\alpha x-\beta(1-x)$. For purpose of brevity we omit the spectral integration over the variables α and β. Note here, that symbols $\bar{\sigma}'s$ denotes the full Lehmann weights defined in (2.1).

The renormalization procedure (2.7) proceeds by direct subtracting of the first two term in Taylor expansion of unrenormalized self-energy (A.12). To make this explicitly we introduce shorthand notation $U(w,l,O)=(l^2-w^2(1-x)x-O+i\epsilon)$. After a little algebra the renormalized quantity corresponding to (A.12) can be evaluated as

$$\begin{aligned}\Pi(\mu;p^2) &= 4i\int\frac{d^4l}{(2\pi)^4}\int_0^1 dxdy\,\times\\ &\times\left[(-\alpha\bar{\sigma}_v(\alpha)\bar{\sigma}_v(\beta)+\right. \qquad (A.13)\\ &+\bar{\sigma}_s(\alpha)\bar{\sigma}_s(\beta))\left(\frac{1}{U^2(p,l,O)}-\frac{1}{U^2(\mu,l,O)}+\frac{2x(1-x)(p^2-\mu^2)}{U^3(\mu,l,O)}\right)+\\ &\left.+\bar{\sigma}_v(\alpha)\bar{\sigma}_v(\beta)\left(\frac{p^2(1-x)}{U^2(p,l,O)}-\frac{\mu^2(1-x)}{U^2(\mu,l,O)}+\frac{2x(1-x)^2\mu^2(p^2-\mu^2)}{U^3(\mu,l,O)}\right)\right].\end{aligned}$$

Matching terms in the first line of (A.13) together we can rewrite them into the familiar DR

$$\frac{4g^2}{(4\pi)^2}\int d\omega\frac{X(\omega;\alpha,\beta)(-\alpha\bar{\sigma}_v(\alpha)\bar{\sigma}_v(\beta)+\bar{\sigma}_s(\alpha)\bar{\sigma}_s(\beta))(p^2-\mu^2)^2}{(\omega-\mu^2)^2(p^2-\omega+i\epsilon)}, \qquad (A.14)$$

where function B was already introduced in (A.9)

The terms in the second brackets of the second line of (A.13) can be matched together in the following fashion:

$$\begin{aligned}&\int_0^1 dy\left[-\frac{2(p^2-\mu^2)^2x(1-x)^2}{(U(p,l,O)y+U(\mu,l,O)(1-y))^3}+\right.\\ &\left.+\left(2x(1-x)^2\mu^2(p^2-\mu^2)\right)\left(\frac{1}{U^3(\mu,l,O)}-\frac{1}{(U(p,l,O)y+U(\mu,l,O)(1-y))^3}\right)\right]=\\ &=\int_0^1 dy\left[-\frac{(p^2-\mu^2)^2(1-x)}{y\left(p^2-\mu^2+\frac{\mu^2}{y}-\frac{O}{x(1-x)y}\right)}+\right.\\ &\left.+\frac{(p^2-\mu^2)^2(1-x)\mu^2}{\left(\mu^2-\frac{O}{x(1-x)y}\right)\left(p^2-\mu^2+\frac{\mu^2}{y}-\frac{O}{x(1-x)y}\right)}\right]. \qquad (A.15)\end{aligned}$$

Adding omitted prefactors, integrating over loop momentum and making the substitution (A.4) in (A.15) gives rise to the DR:

$$\frac{4g_a^2}{(4\pi)^2}\int d\omega \frac{(X(\omega;\alpha,\beta)-X_1(\omega;\alpha,\beta))\omega(p^2-\mu^2)^2}{(\omega-\mu^2)^2(p^2-\omega+i\epsilon)} . \tag{A.16}$$

Putting all together and using relation between X and X_1 we can finally write down the appropriate DR for self-energy of pseudoscalar:

$$\begin{aligned}\Pi(\mu;p^2) &= \int d\omega \frac{\rho_\phi(\omega)(p^2-\mu^2)^2}{(\omega^2-\mu^2)^2(p^2-\omega+i\epsilon)} \\ \rho_\phi(\omega) &= \frac{4g_a^2}{(4\pi)^2}\int d\alpha \int d\beta \Big[\left(-\alpha\bar{\sigma}_v(\alpha)\bar{\sigma}_v(\beta)+\bar{\sigma}_s(\alpha)\bar{\sigma}_s(\beta)\right)X(\omega;\alpha,\beta)+ \\ &\qquad + \omega X_1(\omega;\beta,\alpha)\bar{\sigma}_v(\alpha)\bar{\sigma}_v(\beta)\Big]. \end{aligned} \tag{A.17}$$

For the purpose of numerical solution it is necessary to extract the singular parts of $barSG's$ explicitly. After some trivial manipulation the appropriate formula for ρ_ϕ reads

$$\begin{aligned}\rho_\phi(\omega) = \left(\frac{g}{2\pi}\right)^2 \Bigg[& r_f^2\frac{\omega}{2}\sqrt{1-\frac{4m^2}{\omega}}\Theta(\omega-4m^2)+ \\ & + 2r_f m\int_{(m+m_\phi)^2}^\infty d\beta\left(\frac{1}{2}(\omega-m^2-\beta)\sigma_v(\beta)+m\sigma_s(\beta)\right)X(\omega;m^2,\beta)+ \\ & + \int_{(m+m_\phi)^2}^\infty d\alpha\int_{(m+m_\phi)^2}^\infty d\beta\Bigg(\frac{1}{2}(\omega-\alpha-\beta)\sigma_v(\alpha)\sigma_v(\beta)+ \\ & \qquad + \sigma_s(\alpha)\sigma_s(\beta)\Bigg)X(\omega;\alpha,\beta)\Bigg] ,\end{aligned} \tag{A.18}$$

noting that the first line of (A.18) corresponds with the one loop perturbative contribution.

A.2 Quenched, rainbow, Landau gauge QED

It is shown, that with a massless photon the fermion self-energy function b satisfies DR (2.18) while the function a is exactly zero. In the end of this section we review the results for the case of massive photon. These are used in the numerical study described in the main text.

First we treat massless photon case ($\lambda = 0$), where our approximate electron self-energy

$$\Sigma_0(\not{p}) = ie^2\int \frac{d^4k}{(2\pi)^4}G_0^{\mu\nu}(k)\gamma^\mu\int d\alpha \frac{\bar{\sigma}_v(\alpha)(\not{p}-\not{k})+\bar{\sigma}_s(\alpha)}{(p-k)^2-\alpha+i\epsilon}\gamma^\nu . \tag{A.19}$$

requires one subtraction due to the presence of logarithmic ultraviolet divergence. As it is usual, we fist regularize (A.19), then the subtraction will follow. For this purpose we use the old-fashion Pauli-Villars regularization technique which leads to the following regularized result:

$$\Sigma_\Lambda(\not{p}) = -\frac{e^2}{(4\pi)^2}\int d\alpha \int_0^1 dx \Big[(\bar{\sigma}_v(\alpha)\not{p}(4x-2) + 3\bar{\sigma}_s(\alpha)) \times \\ \times \ln\left(\frac{-p^2x(1-x)+\alpha x+\Lambda^2(1-x)}{-p^2x(1-x)+\alpha x} + i\epsilon\right) + \\ + \frac{2\bar{\sigma}_v(\alpha)\not{p}p^2x^2(1-x)}{p^2x(1-x)+\alpha x}\Big]. \tag{A.20}$$

where Λ represents Pauli-Villars regulator. Making the aforementioned subtraction gives the renormalized functions a, b

$$a(\mu;p^2) = -\int d\alpha \frac{\bar{\sigma}_v(\alpha)e^2}{(4\pi)^2}\Big[\int_0^1 dx(4x-2)\ln\left(\frac{-\mu^2(1-x)+\alpha}{-p^2(1-x)+\alpha} + i\epsilon\right) + \\ + \frac{2p^2}{-p^2(1-x)+\alpha} - \frac{2\mu^2}{-\mu^2(1-x)+\alpha}\Big]$$
$$b(\mu;p^2) = -\int d\alpha \frac{3\bar{\sigma}_s(\alpha)e^2}{(4\pi)^2}\int_0^1 dx \ln\left(\frac{-\mu^2(1-x)+\alpha}{-p^2(1-x)+\alpha} + i\epsilon\right). \tag{A.21}$$

where μ^2 is sme space-like renormalization scale. The per-partes integration of logarithm terms gives

$$a(\mu;p^2) = 0$$
$$b(\mu;p^2) = \int d\alpha \frac{3\bar{\sigma}_s(\alpha)e^2}{(4\pi)^2}\int_0^1 dx \frac{x}{(1-x)}\frac{p^2-\mu^2}{(p^2-\frac{\alpha}{1-x}+i\epsilon)}$$
$$= \left(\frac{e}{4\pi}\right)^2\int d\alpha \int_\alpha^\infty d\omega \frac{3\bar{\sigma}_s(\alpha)(1-\frac{\alpha}{\omega})(p^2-\mu^2)}{(\omega-\mu^2)(p^2-\omega+i\epsilon)}. \tag{A.22}$$

which implies the relation between the absorptive part of self-energy $\pi\rho$ and the Lehmamn function σ:

$$\rho_s(\omega) = -3\left(\frac{e}{4\pi}\right)^2\left[m\left(1-\frac{m^2}{\omega}\right) + \int_{m^2}^\omega d\alpha \sigma_s(\alpha)\left(1-\frac{\alpha}{\omega}\right)\right]. \tag{A.23}$$

Note here, that the Pauli-Villars regularization technique was chosen for convenience only. It is not so difficult to use, for instance, the dimensional regularization technique. Of course, making direct algebraic subtraction of unregularized a, b is also possible. All these purely technically different approaches leads to the same results, that is the subject what we exactly understand under the statement "regularization independence".

When the photon propagator G_o is changed by the introduction of small mass parameter λ

$$G_o^{\mu\nu}(k) = \frac{-g^{\mu\nu}+k^\mu k^\nu/k^2}{k^2-\lambda^2+i\varepsilon} \tag{A.24}$$

then the previous results are slightly modified. Repeating the steps as above it leads to the same form of dispersion relation (2.18) but with different ρ_s,ρ_v, both of them are nonzero now. We get for them

$$
\begin{aligned}
\rho_s(\omega) &= -3\left(\frac{e}{4\pi}\right)^2\left\{mX(\omega,m^2,\lambda^2)+\int d\alpha\sigma_s(\alpha)X(\omega,\alpha,\lambda^2)\right\}\\
\rho_v(\omega) &= \left(\frac{e}{4\pi}\right)^2\left\{X(\omega,m^2,\lambda^2)-2X_1(\omega,m^2,\lambda^2)+\right.\\
&\quad+(\omega-m^2)\frac{X_1(\omega,m^2,\lambda^2)-2X_1(\omega,m^2,0)}{\lambda^2}+\\
&\quad+\int d\alpha\sigma_s(\alpha)\Big[X(\omega,\alpha,\lambda^2)-2X_1(\omega,\alpha,\lambda^2)+\\
&\quad\left.+(\omega-\alpha)\frac{X_1(\omega,\alpha,\lambda^2)-2X_1(\omega,\alpha,0)}{\lambda^2}\Big]\right\}\\
X(\omega,\alpha,\beta) &\equiv \frac{\Delta^{1/2}(\omega,\alpha,\beta)}{\omega}\Theta\left(\omega-(\sqrt{\alpha}+\sqrt{\beta})^2\right)\\
X_1(\omega,\alpha,\beta) &\equiv \frac{\Delta^{1/2}(\omega,\alpha,\beta)[\omega+\beta-\alpha]}{2\omega^2}\Theta\left(\omega-(\sqrt{\alpha}+\sqrt{\beta})^2\right)
\end{aligned}
\tag{A.25}
$$

where Δ is well known triangle function

$$\Delta(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz\,. \tag{A.26}$$

(These functions are already introduced in the appendix A.1, where it is shown how they appear explicitly in our calculation).

Stressed here, that the limit $\lambda\to 0$ can be safely performed an leads to the result (A.23) and $\rho_v=0$.

B. Numerical details

The unitary equations have been solved by the method of numerical iteration. To achieve a reasonable accuracy we should carefully perform the principal value integrations labeled by $[\sigma,\rho]$, $([\sigma,\rho])$ in the equation for fermion (pseudoscalar) weights. For instance the numerical P. integration

$$[\sigma*\rho]=P.\int_{\text{thresh.}}^{\infty}dx\frac{\sigma(s)\rho(x)\frac{s-\mu^2}{x-\mu^2}+\sigma(x)\rho(s)}{s-x}\,. \tag{B.1}$$

is proceed by the following way which is based on the exact relation $0=\int_0^\infty dx\frac{1}{x^2-a^2}$. Therefore we can write for P. integral in (B.1):

$$P.\int_c^\infty dx\frac{f(x)}{a-x}=P.\int_c^\infty dx\frac{f(x)(a+x)-f(a)2a}{a^2-x^2}+f(a)\ln(\frac{a-c}{a+c}) \tag{B.2}$$

where

$$f(x)=\sigma(a)\rho(x)\frac{a-\mu^2}{x-\mu^2}+\sigma(x)\rho(a)\,.$$

The right hand side of the identity (B.2) is particularly useful when evaluated numerically i.e., when $\int\to\Sigma$.

Using some contemporary PC machine the criterion $\sigma^2_{n,n-1} \simeq 10^{-18}$ can be achieved, noting that the typical CPU time is about $10^2 s$ for a grid number of several hundred mesh points. Here, $\sigma^2_{n,n-1}$ represents the error between the solution of n times and $n-1$ times iterated unitary equation i.e.,

$$\sigma^2_{n,n-1} = \frac{\int(\rho_n^2 - \rho_{n-1}^2)}{\int(\rho_n^2 + \rho_{n-1}^2)} . \tag{B.3}$$

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